مباراة الدخول الى معهد الفنون الجميلة للعام 2018-2019

قسم الهندسة المعمارية **مسابقة في الرياضيات (انكليزي)** **المدة: ساعتان**

All CALCULATORS are prohibited.

Exercise I. CALCULUS. ▶ 14 points

Let f and g be two functions defined on $\mathbb{R}^*$ by $f(x) = e^{\frac{-1}{x}}$ and $g(x) = \dfrac{e^{\frac{-1}{x}}}{x^2}$. Denote by (C_f) and (C_g) the representative curves of f and g, respectively. Given $e^{-2} = 0.13$.

Study of g.

1. Determine $\lim\limits_{x\to\pm\infty} g(x)$, $\lim\limits_{x\to 0+} g(x)$ and $\lim\limits_{x\to 0-} g(x)$. Deduce the asymptotes to (C_g).
2. Draw up the variation table of g.
3. Determine the point (x_0, y_0) where g admits a local extremum, and indicate the nature of that extremum.
4. Deduce the position of (C_g) relative to its horizontal asymptote.
5. Plot the curve (C_g).

Study of f.

6. Determine $\lim\limits_{x\to\pm\infty} f(x)$ and $\lim\limits_{x\to 0} f(x)$. Deduce the asymptotes to (C_f).
7. Draw up the variation table of f and determine the position of (C_f) relative to its horizontal asymptote.
8. Plot the curve (C_f).

Area of a domain. Denote by $\mathcal{A}$ the area of the domain bounded by (C_g), the axis $x'Ox$ and the line $x = 0$. Consider, for every integer $n \in \mathbb{N}^*$, the real number $\mathcal{A}_n = \displaystyle\int_0^n g(x)\,dx$.

9. Give a geometric interpretation of $\mathcal{A}_n$.
10. Calculate $\mathcal{A}_n$ and $\lim\limits_{n\to+\infty} \mathcal{A}_n$. Deduce the value of $\mathcal{A}$.

Exercise II. TRUE OR FALSE WITH JUSTIFICATION. ▶ 8+4+8 points

Determine if each proposition is true or false and **justify** your answer. All propositions are independent. *An answer only by true or false, without justification, will not be taken into consideration.*

COMPLEX. Let $z = x + iy = re^{i\theta}$ and $z' = x' + iy' = r'e^{i\theta'}$ be two **non-zero** complex numbers. We define $\varphi(z, z') = z\overline{z'} + \overline{z}z'$.

1. If $|z|^2 = z^2$, then z is a real.
2. If z is pure imaginary, then $|z|^2 = -z^2$.
3. $\varphi(z, z') = 2(xx' + yy')$.
4. $\varphi(z, z') = 2rr'\sin(\theta - \theta')i$.
5. If z is real and z' is pure imaginary, then $\varphi(z, z') = 0$.
6. The set of points $M(z)$ such that $\varphi(z, 1+i) = 2\sqrt{2}$ is the straight line of equation $x + y = \sqrt{2}$.
7. $\varphi(\overline{z}, \frac{1}{z}) = 2$.
8. The set of points $M(z)$ such that $\varphi(z, z-2) = 2$ is a circle $\mathcal{C}\big((1,0); R = \sqrt{2}\big)$.

PROBABILITY. An urn contains 8 balls (3 red and 5 black) and 6 cubes (2 red and 4 black). Two objects are drawn simultaneously, assuming the drawings are equiprobable. **Two objects are said to be identical if they have the same shape and the same color.**

9. The probability of drawing a cube and a ball of different colors is $\frac{22}{91}$.
10. The probability of drawing a cube and a ball of the same color is $\frac{69}{91}$.
11. The probability of drawing two identical objects is $\frac{20}{91}$.
12. The probability of drawing at least a black ball is $\frac{45}{91}$.

GEOMETRY. In the space referred to a direct orthonormal basis $(O, \vec{i}, \vec{j}, \vec{k})$, given :

the planes :

- ▷ (P_θ) of equation $y - x = \sin^2\theta\,;\ \theta \in \mathbb{R}$,
- ▷ (Q_θ) of equation $z - y = \cos^2\theta\,;\ \theta \in \mathbb{R}$,
- ▷ (R) of equation $x - y - 2z + 1 = 0$,

and the straight lines :

- ▷ (E) of equation $x = 1 + \lambda$, $y = -2 - \lambda$ and $z = 4 + 3\lambda\,;\ \lambda \in \mathbb{R}$,
- ▷ (Δ_θ) is the intersection line of the planes (P_θ) and (Q_θ).

13. The line (E) passes through the point $I(1, -1, 3)$.
14. (E) meets (R) in one and only one point.
15. For all θ, (Δ_θ) is contained in the plane of equation $z - x - 1 = 0$.
16. For all θ, (Δ_θ) passes through the point $A(-1, -\cos^2\theta, 0)$.
17. The plane of equation $3x + 2y + z - 5 = 0$ is parallel to (R).
18. The plane that contains (E) and perpendicular to (R) is of equation $x + y + 1 = 0$.
19. For all θ, the vector $(1, -1, 0)$ is orthogonal to (Δ_θ).
20. For all θ, (Δ_θ) is perpendicular to the plane of equation $x + y + z - 3 = 0$.

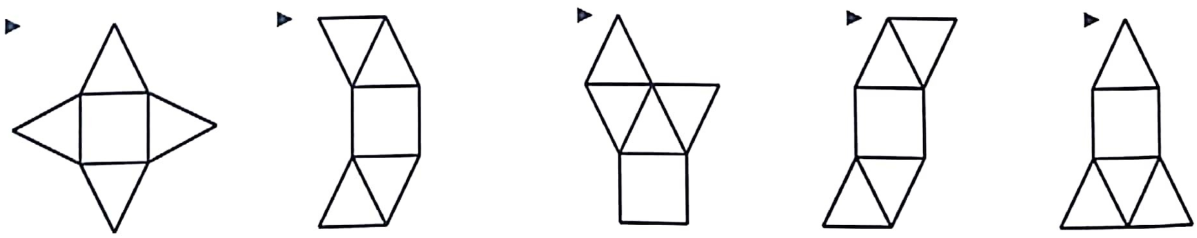
Exercise III. **CHOICE OF A GOOD ANSWER WITHOUT JUSTIFICATION.** ▶ 6 points

In the following exercise, **only one** answer among the proposed answers is correct. *Write dawn the number of each question, then write or trace the answer that corresponds to it, without any justification.*

Q1. If $x^2 - 4x + 2 = 0$, then $x + \dfrac{2}{x}$ equals to

▶ -4 ▶ -2 ▶ 0 ▶ 2 ▶ 4

Q2. Which one of the five proposed patterns does not compose a pyramid?



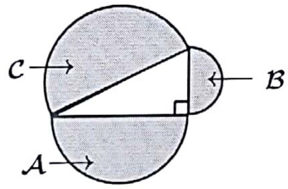
Q3. How much is $\dfrac{2^{2014} - 2^{2013}}{2^{2013} - 2^{2012}}$?

▶ 2^{2011} ▶ 2^{2012} ▶ 2^{2013} ▶ 1 ▶ 2

Q4. The figure shows in gray three semi-disks, the diameters of which are respectively the three sides of a right triangle. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the areas, in cm^2, of these three semi-disks, which relationship among the following ones is necessarily true?

▶ $\mathcal{A} + \mathcal{B} < \mathcal{C}$ ▶ $\mathcal{A} + \mathcal{B} = \mathcal{C}$ ▶ $\sqrt{\mathcal{A}} + \sqrt{\mathcal{B}} = \sqrt{\mathcal{C}}$

▶ $\mathcal{A}^2 + \mathcal{B}^2 = \mathcal{C}^2$ ▶ $\mathcal{A}^2 + \mathcal{B}^2 = \mathcal{C}$



Q5. In the plane referred to a direct orthonormal basis, we consider the curves of the following functions $f(x) = 2 - x^2$ and $g(x) = x^2 - 1$. Into how many regions do these two curves split the plane?

▶ 3 ▶ 4 ▶ 5 ▶ 6 ▶ 7

Q6. Two identical circular cylinders (of volume V_1 each) are cut along a vertical dotted line and glued together to form a singular circular cylinder of volume V_2 (see figure). What is the ratio $\frac{V_2}{V_1}$ of the volume of the large cylinder to the volume of one of the initial cylinders?

▶ 2 ▶ 3 ▶ π

▶ 4 ▶ 8