# The Basic Theory for Partial Functional Differential Equations and Applications

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## 1 Introduction

Let $(X, |\cdot|)$ be an infinite dimensional Banach space and $\mathcal{L}(X)$ be the space of bounded linear operators from $X$ into $X$. Suppose that $r > 0$ is a given real number. $\mathcal{C}([-r, 0], X)$ denotes the space of continuous functions from $[-r, 0]$ to $X$ with the uniform convergence topology and we will use simply $\mathcal{C}_X$ for $\mathcal{C}([-r, 0], X)$. For $u \in \mathcal{C}([-r, b], X)$, $b > 0$ and $t \in [0, b]$, let $u_t$ denote the element of $\mathcal{C}_X$ defined by $u_t(\theta) = u(t + \theta)$, $-r \leq \theta \leq 0$.

By an abstract semilinear functional differential equation on the space $X$, we mean an evolution equation of the type

$$\begin{cases} \dfrac{du}{dt}(t) = A_0 u(t) + F(t, u_t), \quad t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{1}$$

where $A_0 : D(A_0) \subseteq X \to X$ is a linear operator, $F$ is a function from $[0, +\infty) \times \mathcal{C}_X$ into $X$ and $\varphi \in \mathcal{C}_X$ are given. The initial value problem associated with (1) is the following : given $\varphi \in \mathcal{C}_X$, to find a continuous

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function $u : [-r, h) \to X,\ h > 0$, differentiable on $[0, h)$ such that $u(t) \in D(A_0)$, for $t \in [0, h)$ and $u$ satisfies the evolution equation of (1) for $t \in [0, h)$ and $u_0 = \varphi$.

It is well-known (see for example [65] and [67]) that the classical semigroup theory ensures the well posedness of Problem (1) when $A_0$ is the infinitesimal generator of a $C_0$-semigroup of bounded linear operators $(T_0(t))_{t\geq 0}$ in $X$ or, equivalently when

**(i)** $\overline{D(A_0)} = X$,

**(ii)** there exist $M_0, \omega_0 \in I\!R$ such that if $\lambda > \omega_0 \quad (\lambda I - A_0)^{-1} \in \mathcal{L}(X)$ and

$$\left\| (\lambda - \omega_0)^n (\lambda I - A_0)^{-n} \right\| \leq M_0, \qquad \forall\, n \in \mathbb{N}.$$

In this case one can prove existence and uniqueness of a solution of (1), for example by using the variation-of-constants formula ([65], [67])

$$u(t) = \begin{cases} T_0(t)\varphi(0) + \int_0^t T_0(t-s)F(s, u_s)ds, & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \in [-r, 0], \end{cases}$$

for every $\varphi \in \mathcal{C}_X$.

For related results, see for example Travis and Webb [65], Webb [66], Fitzgibbon [43], Kunish and Schappacher ([49], [50]), Memory ([53], [54]), Wu [67], and the references therein. In all of the quoted papers, $A_0$ is an operator verifying **(i)** and **(ii)**. In the applications, it is sometimes convenient to take initial functions with more restrictions. There are many examples in concrete situations where evolution equations are not densely defined. Only hypothesis (**ii**) holds. One can refer for this to [40] for more details. Non-density occurs, in many situations, from restrictions made on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or from boundary conditions (e.g., the space $C^1$ with null value on the boundary is non-dense in the space of continuous functions). Let us now briefly discuss the use of integrated semigroups. In the case where the mapping $F$ in Equation (1) is equal to zero, the problem can still be handled by using the classical semigroups theory because $A_0$ generates a strongly continuous semigroup in the space $\overline{D(A_0)}$. But, if $F \neq 0$, it is necessary to impose additional restrictions. A case which is easily handled is when $F$ takes their values in $\overline{D(A_0)}$. On the other hand, the integrated semigroups theory allows the range of the operators $F$ to be any subset of $X$.

**Example 1** *Consider the model of population dynamics with delay described by*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,a) + \dfrac{\partial u}{\partial a}(t,a) = f(t,a,u_t(\cdot,a)), & (t,a) \in [0,T], \times [0,l], \\ u(t,0) = 0, & t \in [0,T], \\ u(\theta,a) = \varphi(\theta,a), & (\theta,a) \in [-r,0] \times [0,l], \end{cases} \tag{2}$$

*where $\varphi$ is a given function on $\mathcal{C}_X := \mathcal{C}([-r,0],X)$, with $X = \mathcal{C}([0,l], I\!R)$. By setting $V(t) = u(t,\cdot)$, we can reformulate the partial differential problem (2) as an abstract semilinear functional differential equation*

$$\begin{cases} V'(t) = A_0V(t) + F(t,V_t), & t \in [0,T], \\ V_0 = \varphi \in \mathcal{C}_X, \end{cases} \tag{3}$$

*where*

$$\begin{cases} D(A_0) = \left\{ u \in \mathcal{C}^1([0,l], I\!R); \quad u(0) = 0 \right\}, \\ A_0u = -u', \end{cases}$$

*and $F : [0,T] \times \mathcal{C}_X \to X$ is defined by $F(t,\varphi)(a) = f(t,a,\varphi(\cdot,a))$ for $t \in [0,T]$, $\varphi \in \mathcal{C}_X$ and $a \in [0,l]$.*

**Example 2** *Consider the following ecological model taken from [67] and it's described by the following reaction-diffusion equation with delay:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(t,x,u_t(\cdot,x)), & t \in [0,T], \ x \in \Omega, \\ u(t,x) = 0, & t \in [0,T], \ x \in \partial\Omega, \\ u(\theta,x) = \varphi(\theta,x), & \theta \in [-r,0], \ x \in \Omega, \end{cases} \tag{4}$$

*where $\Omega \subset I\!R^n$ is a bounded open set with regular boundary $\partial\Omega$, $\Delta$ is the Laplace operator in the sense of distributions on $\Omega$ and $\varphi$ is a given function on $\mathrm{C}_X := \mathrm{C}([-r,0],X)$, with $X = \mathcal{C}\left(\overline{\Omega}, I\!R\right)$.*
*The problem (4) can be reformulated as the abstract semilinear functional differential equation (3), with*

$$\begin{cases} D(A_0) = \left\{ u \in \mathcal{C}\left(\overline{\Omega}, I\!R\right); \quad \Delta u \in \mathcal{C}\left(\overline{\Omega}, I\!R\right) \quad \text{and } u = 0 \quad \text{on } \partial\Omega \right\}, \\ A_0u = \Delta u, \end{cases}$$

*and $F : [0,T] \times \mathcal{C}_X \to X$ is defined by $F(t,\varphi)(x) = f(t,x,\varphi(\cdot,x))$ for $t \in [0,T]$, $\varphi \in \mathcal{C}_X$ and $x \in \Omega$.*

In the two examples given here, the operator $A_0$ satisfies **(ii)** but the domain $D(A_0)$ is not dense in $X$ and so, $A_0$ does not generate a $C_0-$semigroup. Of course, there are many other examples encountered in the applications in which the operator $A_0$ satisfies only **(ii)** (see [40]).

We will study here the abstract semilinear functional differential equation (1) in the case when the operator $A_0$ satisfies only the Hille-Yosida condition (**ii**). After providing some background materials in Section 2, we proceed to establish the main results. A natural generalized notion of solutions is provided in Section 3 by integral solutions. We derive a variation-of-constants formula which allows us to transform the integral solutions of the general equation to solutions of an abstract Volterra integral equation. We prove the existence, uniqueness, regularity and continuous dependence on the initial condition. These results give natural generalizations of results in [65] and [67]. In Section 4, we consider the autonomous case. We prove that the solutions generate a nonlinear strongly continuous semigroup, which satisfies a compactness property. In the linear case, the solutions are shown to generate a locally Lipschitz continuous integrated semigroup. In Section 5, we use a principle of linearized stability for strongly continuous semigroups given by Desh and Schappacher [41] (see also [58], [59] and [61]) to study, in the nonlinear autonomous case, the stability of Equation (1). In Section 6, we show in the linear autonomous case, the existence of a direct sum decomposition of a state space into three subspaces : stable, unstable and center, which are semigroup invariants. As a consequence of the results established in Section 6, the existence of bounded, periodic and almost periodic solutions is established in the sections 7 and 8. In the end, we give some examples.

## 2 Basic results

In this section, we give a short review of the theory of integrated semigroups and differential operators with non-dense domain. We start with a few definitions.

**Definition 3** *[34] Let $X$ be a Banach space. A family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called an integrated semigroup if the following conditions are satisfied :*
**(i)** $S(0) = 0$*;*
**(ii)** *for any $x \in X$, $S(t)x$ is a continuous function of $t \geq 0$ with values in $X$;*
**(iii)** *for any $t, s \geq 0$ $\quad S(s)S(t) =_0^s (S(t+\tau) - S(\tau))d\tau$.*

**Definition 4** *[34] An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $M \geq 0$ and $\omega \in I\!R$ such that*

$$\|S(t)\| \leq Me^{\omega t} \quad \textit{for } t \geq 0.$$

*Moreover $(S(t))_{t\geq 0}$ is called non-degenerate if $S(t)x = 0$, for all $t \geq 0$, implies that $x = 0$.*

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda) := \lambda_0^{+\infty} e^{-\lambda t} S(t)dt$ exists for all $\lambda$ with $\mathcal{R}e(\lambda) > \omega$. $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. $R(\lambda)$ satisfies the following expression

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu),$$

and in the case when $(S(t))_{t\geq 0}$ is non-degenerate, there exists a unique operator $A$ satisfying $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of $A$) such that

$$R(\lambda) = (\lambda I - A)^{-1}, \text{ for all } \mathcal{R}e(\lambda) > \omega.$$

This operator $A$ is called the generator of $(S(t))_{t\geq 0}$.
We have the following definition.

**Definition 5** *[34] An operator $A$ is called a generator of an integrated semigroup, if there exists $\omega \in I\!R$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda_0^{+\infty} e^{-\lambda t} S(t)dt$, for all $\lambda > \omega$.*

**Remark 1** *If an operator $A$ is the generator of an integrated semigroup $(S(t))_{t\geq 0}$, then $\forall \lambda \in I\!R$, $A - \lambda I$ is the generator of the integrated semigroup $(S_\lambda(t))_{t\geq 0}$ given by*

$$S_\lambda(t) = e^{-\lambda t} S(t) + \lambda \int_0^t e^{-\lambda s} S(s)\, ds.$$

**Proposition 6** *[34] Let $A$ be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A) \textit{ and } S(t)x = A\left(\int_0^t S(s)xds\right) + tx.$$

*Moreover, for all* $x \in D(A),\ t \geq 0$

$$S(t)x \in D(A) \qquad \text{and} \qquad AS(t)x = S(t)Ax,$$

*and*

$$S(t)x = tx + \int_0^t S(s)Ax\, ds.$$

**Corollary 7** *[34] Let* $A$ *be the generator of an integrated semigroup* $(S(t))_{t\geq 0}$*. Then for all* $x \in X$ *and* $t \geq 0$ *one has* $S(t)x \in \overline{D(A)}$*.*
*Moreover, for* $x \in X$*,* $S(\cdot)x$ *is right-sided differentiable in* $t \geq 0$ *if and only if* $S(t)x \in D(A)$*. In that case*

$$S'(t)x = AS(t)x + x.$$

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

**Definition 8** *[52] An integrated semigroup* $(S(t))_{t\geq 0}$ *is called locally Lipschitz continuous, if for all* $\tau > 0$ *there exists a constant* $k(\tau) > 0$ *such that*

$$\|S(t) - S(s)\| \leq k(\tau)\, |t - s|\,, \text{ for all } t, s \in [0, \tau]\,.$$

In this case, we know from [52], that $(S(t))_{t\geq 0}$ is exponentially bounded.

**Definition 9** *[52] We say that a linear operator* $A$ *satisfies the Hille-Yosida condition* $(\mathbf{HY})$ *if there exist* $M \geq 0$ *and* $\omega \in I\!R$ *such that* $(\omega, +\infty) \subset \rho(A)$ *and*

$$\sup\left\{(\lambda - \omega)^n \left\|(\lambda I - A)^{-n}\right\|,\ n \in \mathbb{N},\ \lambda > \omega\right\} \leq M. \qquad (\mathbf{HY})$$

The following theorem shows that the Hille-Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

**Theorem 10** *[52] The following assertions are equivalent.*
**(i)** *$A$ is the generator of a locally Lipschitz continuous integrated semigroup,*
**(ii)** *$A$ satisfies the condition* $(\mathbf{HY})$*.*

In the sequel, we give some results for the existence of solutions of the following Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + f(t), \quad t \geq 0, \\ u(0) = x \in X, \end{cases} \tag{5}$$

where $A$ satisfies the condition (**HY**), without being densely defined.

By a solution of Problem (5) on $[0,T]$ where $T>0$, we understand a function $u \in \mathcal{C}^1([0,T],X)$ satisfying $u(t) \in D(A)$ $(t \in [0,T])$ such that the two relations in (5) hold.

The following result is due to Da Prato and Sinestrari.

**Theorem 11** *[40] Let $A : D(A) \subseteq X \to X$ be a linear operator, $f : [0,T] \to X$, $x \in D(A)$ such that*
**(a)** *$A$ satisfies the condition* (**HY**).
**(b)** *$f(t) = f(0) + \int_0^t g(s)\,ds$ for some Bochner-integrable function $g$.*
**(c)** *$Ax + f(0) \in \overline{D(A)}$.*
*Then there exists a unique solution $u$ of Problem (5) on the interval $[0,T]$, and for each $t \in [0,T]$*

$$|u(t)| \leq Me^{\omega t}\left(|x| + \int_0^t e^{-\omega s}\,|f(s)|\,ds\right).$$

In the case where $x$ is not sufficiently regular (that is, $x$ is just in $\overline{D(A)}$) there may not exist a strong solution $u(t) \in X$ but, following the work of Da Prato and Sinestrari [40], Problem (5) may still have an integral solution.

**Definition 12** *[40] Given $f \in L^1_{loc}(0,+\infty;X)$ and $x \in X$, we say that $u : [0,+\infty) \to X$ is an integral solution of (5) if the following assertions are true*
**(i)** *$u \in \mathcal{C}([0,+\infty);X)$,*
**(ii)** *${}_0^t u(s)ds \in D(A)$, for $t \geq 0$,*
**(iii)** *$u(t) = x + A{}_0^t u(s)ds + {}_0^t f(s)ds$, for $t \geq 0$.*

From this definition, we deduce that for an integral solution $u$, we have $u(t) \in \overline{D(A)}$, for all $t>0$, because $u(t) = h \to 0 \lim \frac{1}{h}{}_t^{t+h} u(s)ds$ and ${}_t^{t+h} u(s)ds \in D(A)$. In particular, $x \in \overline{D(A)}$ is a necessary condition for the existence of an integral solution of (5).

**Theorem 13** *[37] Suppose that $A$ satisfies the condition* (**HY**)*, $x \in \overline{D(A)}$ and $f : [0,+\infty) \to X$ is a continuous function. Then the problem (5) has a unique integral solution which is given by*

$$u(t) = S'(t)x + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \quad \textit{pour } t \geq 0,$$

*where $S(t)$ is the integrated semigroup generated by $A$.*
*Furthermore, the function $u$ satisfies the inequality*

$$|u(t)| \leq Me^{\omega t}\left(|x| + \int_0^t e^{-\omega s}|f(s)|\,ds\right), \text{ for } t \geq 0.$$

Note that Theorem 13 also says that $\int_0^t S(t-s)f(s)ds$ is differentiable with respect to $t$.

## 3 Existence, uniqueness and regularity of solutions

We restate the problem

$$\begin{cases} \dfrac{du}{dt}(t) = A_0 u(t) + F(t, u_t), & t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_X, \end{cases} \tag{1}$$

where $F : [0,T] \times \mathcal{C}_X \to X$ is a continuous function.
Throughout this work, we assume that $A_0$ satisfies the Hille-Yosida condition on $X$ :
(**HY**) there exist $M_0 \geq 0$ and $\omega_0 \in I\!R$ such that $(\omega_0, +\infty) \subset \rho(A_0)$ and

$$\sup\left\{(\lambda - \omega_0)^n \left\|(\lambda I - A_0)^{-n}\right\|,\, n \in \mathbb{N},\, \lambda > \omega_0\right\} \leq M_0.$$

We know from Theorem 10 that $A_0$ is the generator of a locally Lipschitz continuous integrated semigroup $(S_0(t))_{t\geq 0}$ on $X$, (and $(S_0(t))_{t\geq 0}$ is exponentially bounded).

In view of the remark following Definition 5, we will sometimes assume without loss of generality that $\omega_0 = 0$.

Consider first the linear Cauchy problem

$$\begin{cases} u'(t) = A_0 u(t), & t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_X. \end{cases}$$

This problem can be reformulated as a special case of an abstract semilinear functional differential equation with delay. This is

$$\begin{cases} u'(t) = (Au_t)(0), & t \geq 0, \\ u_0 = \varphi, \end{cases}$$

where

$$\begin{cases} D(A) = \left\{\varphi \in \mathcal{C}^1([-r,0], X);\quad \varphi(0) \in D(A_0), \quad \varphi'(0) = A_0\varphi(0)\right\}, \\ A\varphi = \varphi'. \end{cases}$$

We can show, by using the next result and Theorem 10, that the operator $A$ satisfies the condition (**HY**).

**Proposition 14** *The operator $A$ is the generator of a locally Lipschitz continuous integrated semigroup on* $C_X$ *given by*

$$(S(t)\varphi)(\theta) = \begin{cases} \int_\theta^0 \varphi(s)\,ds + S_0(t+\theta)\varphi(0), & t+\theta \geq 0, \\ \int_\theta^{t+\theta} \varphi(s)\,ds, & t+\theta < 0. \end{cases}$$

*for* $t \geq 0$, $\theta \in [-r, 0]$ *and* $\varphi \in C_X$.

**Proof.** It is easy to see that $(S(t))_{t\geq 0}$ is an integrated semigroup on $C_X$.

Consider $\tau > 0$, $t, s \in [0, \tau]$ and $\varphi \in C_X$.

If $t+\theta \geq 0$ and $s+\theta \geq 0$, we have

$$(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta) = S_0(t+\theta)\varphi(0) - S_0(s+\theta)\varphi(0).$$

It follows immediately that there exists a constant $k := k(\tau) > 0$ such that

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| \leq k\,|t-s|\,|\varphi(0)|\,,$$

because $(S_0(t))_{t\geq 0}$ is Lipschitz continuous on $[0, \tau]$.

If $t+\theta \leq 0$ and $s+\theta \leq 0$, we have

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| = 0.$$

If $t+\theta \geq 0$ and $s+\theta \leq 0$, we obtain

$$(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta) = S_0(t+\theta)\varphi(0) + \int_{s+\theta}^0 \varphi(u)du,$$

then

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| \leq k\,(t+\theta)\,|\varphi(0)| - (s+\theta)\,\|\varphi\|\,,$$

This implies that

$$\|S(t) - S(s)\| \leq (k+1)\,|t-s|\,.$$

It may be concluded that $(S(t))_{t\geq 0}$ is locally Lipschitz continuous.

In order to prove that $A$ is the generator of $(S(t))_{t\geq 0}$, we calculate the spectrum and the resolvent operator of $A$.

Consider the equation

$$(\lambda I - A)\,\varphi = \psi,$$

where $\psi$ is given in $\mathcal{C}_X$ , and we are looking for $\varphi \in D(A)$. The above equation reads

$$\lambda\varphi(\theta) - \varphi'(\theta) = \psi(\theta), \qquad \theta \in [-r, 0]\,.$$

Whose solutions are such that

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) + \int_\theta^0 e^{\lambda(\theta-s)}\psi(s)\,ds, \quad \theta \in [-r, 0]\,.$$

$\varphi$ is in $D(A)$ if $\varphi(0) \in D(A_0)$ and $\varphi'(0) = A_0\varphi(0)$, that is

$$\varphi(0) \in D(A_0) \qquad \text{and} \quad (\lambda I - A_0)\,\varphi(0) = \psi(0).$$

By assumption on $A_0$, we know that $(0, +\infty) \subset \rho(A_0)$. So, for $\lambda > 0$, the above equation has a solution $\varphi(0) = (\lambda I - A_0)^{-1}\,\psi(0)$.

Therefore, $(0, +\infty) \subset \rho(A)$ and

$$\left((\lambda I - A)^{-1}\,\psi\right)(\theta) = e^{\lambda\theta}\,(\lambda I - A_0)^{-1}\,\psi(0) + \int_\theta^0 e^{\lambda(\theta-s)}\psi(s)ds,$$

for $\theta \in [-r, 0]$ and $\lambda > 0$.

On the other hand, from the formula stated in Proposition 14, it is clear that $t \to (S(t)\varphi)\,(\theta)$ has at most exponential growth, not larger than $\omega_0 = 0$. Therefore, one can defined its Laplace transform, for each $\lambda > 0$. We obtain

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,(S\,(t)\,\varphi)\,(\theta)\;dt \;&= \int_0^{-\theta} e^{-\lambda t}\int_\theta^{t+\theta} \varphi(s)ds\,dt\, + \int_{-\theta}^{+\infty} e^{-\lambda t}\int_\theta^0 \varphi(s)ds\,dt\\ &\quad+ \int_{-\theta}^{+\infty} e^{-\lambda t}S_0(t+\theta)\varphi(0)\,dt.\end{aligned}$$

Integrating by parts the first expression, it yields

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,(S\,(t)\,\varphi)\,(\theta)\;dt \;&= -\tfrac{e^{\lambda\theta}}{\lambda}\int_\theta^0 \varphi(s)ds + \tfrac{1}{\lambda}\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)ds\\ &\quad+\tfrac{e^{\lambda\theta}}{\lambda}\int_\theta^0 \varphi(s)ds + \int_0^{+\infty} e^{\lambda(\theta-s)}S_0(s)\varphi(0)ds,\\ &= \tfrac{e^{\lambda\theta}}{\lambda}\left(\int_\theta^0 e^{-\lambda s}\varphi(s)ds + \int_0^{+\infty} e^{-\lambda s}S_0(s)\varphi(0)ds\right),\\ &= \tfrac{e^{\lambda\theta}}{\lambda}\left(\int_\theta^0 e^{-\lambda s}\varphi(s)ds + (\lambda I - A_0)^{-1}\,\varphi(0)\right),\\ &= \frac{1}{\lambda}\left((\lambda I - A)^{-1}\,\varphi\right)(\theta).\end{aligned}$$

So, $A$ is related to $(S(t))_{t\geq 0}$ by the formula which characterizes the infinitesimal generator of an integrated semigroup. The proof of Proposition 14 is complete ■

Our next objective is to construct an integrated version of Problem (1) using integrated semigroups. We need to extend the integrated semigroup $(S(t))_{t\geq 0}$ to the space $\widetilde{\mathcal{C}}_X = \mathcal{C}_X \oplus \langle X_0 \rangle$, where
$\langle X_0 \rangle = \{X_0 c, \;\; c \in X \text{ and } \; (X_0 c)(\theta) = X_0(\theta)c\}$ and $X_0$ denotes the function defined by $X_0(\theta) = 0$ if $\theta < 0$ and $X_0(0) = Id_X$. We shall prove that this extension determines a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathcal{C}}_X$.

**Proposition 15** *The family of operators $(\widetilde{S}(t))_{t\geq 0}$ defined on $\widetilde{\mathcal{C}}_X$ by*

$$\widetilde{S}(t)\varphi = S(t)\varphi, \;\; \text{for } \varphi \in \mathcal{C}_X$$

*and*

$$\left(\widetilde{S}(t)X_0 c\right)(\theta) = \begin{cases} S_0(t+\theta)c, & \text{if } t+\theta \geq 0, \\ 0, & \text{if } t+\theta \leq 0, \quad \text{for } c \in X, \end{cases}$$

*is a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathcal{C}}_X$ generated by the operator $\widetilde{A}$ defined by*

$$\begin{cases} D(\widetilde{A}) = \left\{\varphi \in \mathcal{C}^1([-r,0], X); \;\; \varphi(0) \in D(A_0)\right\}, \\ \widetilde{A}\varphi = \varphi' + X_0\left(A_0\varphi(0) - \varphi'(0)\right). \end{cases}$$

**Proof.** Using the same reasoning as in the proof of Proposition 14, one can show that $(\widetilde{S}(t))_{t\geq 0}$ is a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathcal{C}}_X$.

The proof will be completed by showing that

$$\begin{cases} (0,+\infty) \subset \rho(\widetilde{A}) \text{ and} \\ \left(\lambda I - \widetilde{A}\right)^{-1}\widetilde{\varphi} = \lambda \int_0^{+\infty} e^{-\lambda t}\widetilde{S}(t)\widetilde{\varphi}\, dt, \text{ for } \lambda > 0 \text{ and } \widetilde{\varphi} \in \widetilde{\mathcal{C}}_X. \end{cases}$$

For this, we need the following lemma.

**Lemma 16** *For $\lambda > 0$, one has*
*(i)* $D(\widetilde{A}) = D(A) \oplus \langle e^{\lambda \cdot} \rangle$, *where* $\langle e^{\lambda \cdot} \rangle = \left\{e^{\lambda \cdot}c; \;\; c \in D(A_0), \, (e^{\lambda \cdot}c)(\theta) = e^{\lambda\theta}c\right\}$,
*(ii)* $(0,+\infty) \subset \rho(\widetilde{A})$ *and*

$$(\lambda I - \widetilde{A})^{-1}(\varphi + X_0 c) = (\lambda I - A)^{-1}\varphi + e^{\lambda \cdot}(\lambda I - A_0)^{-1}c,$$

*for every* $(\varphi, c) \in \mathcal{C}_X \times X$ .

**Proof of the lemma.** For the proof of (i), we consider the following operator

$$\begin{aligned} l : D(\widetilde{A}) &\to X \\ \varphi &\to l(\varphi) = A_0\varphi(0) - \varphi'(0). \end{aligned}$$

Let $\widetilde{\Psi} \in D(\widetilde{A})$ and $\lambda > 0$. Setting $\Psi = \widetilde{\Psi} - e^{\lambda \cdot}(\lambda I - A_0)^{-1} l(\widetilde{\Psi})$, we deduce that $\Psi \in Ker(l) = D(A)$, and the decomposition is clearly unique.

(ii) Consider the equation

$$\left(\lambda I - \widetilde{A}\right)\left(\varphi + e^{\lambda \cdot} c\right) = \psi + X_0 a,$$

where $(\psi, a)$ is given, $\psi \in \mathcal{C}_X,\ a \in X$, and we are looking for $(\varphi,\, c)$, $\varphi \in D(A),\ \ c \in D(A_0)$.

This yields

$$(\lambda I - A)\,\varphi + \lambda e^{\lambda \cdot} c - \lambda e^{\lambda \cdot} c - X_0\,(A_0 c - \lambda c) = \psi + X_0 a,$$

which has the solution

$$\begin{cases} \varphi = (\lambda I - A)^{-1}\,\psi, \\ c = (\lambda I - A_0)^{-1}\,a. \end{cases}$$

Consequently,

$$\begin{cases} (0, +\infty) \subset \rho(\widetilde{A}), \\ \left(\lambda I - \widetilde{A}\right)^{-1}(\varphi + X_0 c) = (\lambda I - A)^{-1}\,\varphi + e^{\lambda \cdot}\,(\lambda I - A_0)^{-1}\,c. \end{cases}$$

This complete the proof of the lemma ■

We now turn to the proof of the proposition. All we want to show is that $\dfrac{1}{\lambda}\left(\lambda I - \widetilde{A}\right)^{-1}\widetilde{\varphi}$ can be expressed as the Laplace transform of $\widetilde{S}(t)\widetilde{\varphi}$. In view of the decomposition and what has been already done for $A$, we may restrict our attention to the case when $\widetilde{\varphi} = X_0 c$. In this case, we have

$$\begin{aligned} \left((\lambda I - \widetilde{A})^{-1} X_0 c\right)(\theta) \ &= e^{\lambda\theta}\,(\lambda I - A_0)^{-1}\,c, \\ &= \lambda e^{\lambda\theta} \int_0^{+\infty} e^{-\lambda t} S_0(t) c\, dt, \\ &= \lambda \int_{-\theta}^{+\infty} e^{-\lambda t} S_0(t + \theta) c\, dt, \\ &= \lambda \int_0^{+\infty} e^{-\lambda t}\left(\widetilde{S}(t) X_0 c\right)(\theta)\ dt, \end{aligned}$$

which completes the proof of the proposition ■

We will need also the following general lemma.

**Lemma 17** *Let $(U(t))_{t\geq 0}$ be a locally Lipschitz continuous integrated semigroup on a Banach space $(E, |\cdot|)$ generated by $(A, D(A))$ and $G : [0,T] \to E$ $(0 < T)$, a Bochner-integrable function. Then, the function $K : [0,T] \to E$ defined by*

$$K(t) = \int_0^t U(t-s)G(s)\, ds$$

*is continuously differentiable on $[0,T]$ and satisfies, for $t \in [0,T]$,*

**(i)** $K'(t) = h \searrow 0\lim \frac{1}{h} \int_0^t U'(t-s)U(h)G(s)\, ds = \lambda \to +\infty\lim \int_0^t U'(t-s)(A_\lambda G(s))\, ds$,

*with* $A_\lambda = \lambda \left(\lambda I - A\right)^{-1}$,

**(ii)** $K(t) \in D(A)$,

**(iii)** $K'(t) = AK(t) + \int_0^t G(s)\, ds$.

**Proof.** Theorem 13 implies that $K$ is continuously differentiable on $[0,T]$, and for the prove of **(ii)** and **(iii),** see [37], [47] and [63]. On the other hand, we know by the definition of an integrated semigroup that

$$U(t)U(h)x = \int_0^t \left(U(s+h)x - U(s)x\right)\, ds,$$

for $t, h \geq 0$ and $x \in E$. This yields that the function $t \to U(t)U(h)x$ is continuously differentiable on $[0,T]$, for each $h \geq 0$; $x \in E$, and satisfies

$$U'(t)U(h)x = U(t+h)x - U(t)x.$$

Furthermore, we have

$$\begin{aligned} K'(t) &= h \searrow 0\lim \left( \frac{1}{h} \int_0^t \left(U(t+h-s) - U(t-s)\right) G(s)\, ds \right. \\ &\qquad \left. + \frac{1}{h} \int_t^{t+h} U(t+h-s)G(s)\, ds \right). \end{aligned}$$

If we put, in the second integral of the right-hand side, $u = \frac{1}{h}(s-t)$, we obtain

$$\frac{1}{h} \int_t^{t+h} U(t+h-s)G(s)\, ds = \int_0^1 U(h(1-u))G(t+hu)\, du.$$

This implies that

$$h \searrow 0\lim \frac{1}{h} \int_t^{t+h} U(t+h-s)G(s)\, ds = 0.$$

Hence

$$K'(t) = h \searrow 0\lim \frac{1}{h} \int_0^t \left( U(t+h-s) - U(t-s) \right) G(s)\, ds.$$

But

$$U(t+h-s) - U(t-s) = U'(t-s)U(h).$$

It follows that, for $t \in [0,T]$

$$K'(t) = h \searrow 0\lim \int_0^t U'(t-s)\frac{1}{h}U(h)G(s)\, ds.$$

For the prove of the last equality of (i), see [64] ■

We are now able to state a first result of existence and uniqueness of solutions.

**Theorem 18** *Let $F : [0,T] \times \mathcal{C}_X \to X$ be continuous and satisfy a Lipschitz condition*

$$|F(t,\varphi) - F(t,\psi)| \leq L \|\varphi - \psi\|, \quad t \in [0,T] \ \text{and}\ \varphi, \psi \in \mathcal{C}_X,$$

*where $L$ is a positive constant. Then, for given $\varphi \in \mathcal{C}_X$, such that $\varphi(0) \in \overline{D(A_0)}$, there exists a unique function $y : [0,T] \to \mathcal{C}_X$ which solves the following abstract integral equation*

$$y(t) = S'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(s,y(s))\, ds, \quad \text{for } t \in [0,T], \qquad (6)$$

*with $(S(t))_{t\geq 0}$ given in Proposition 14 and $\left(\widetilde{S}(t)\right)_{t\geq 0}$ in Proposition 15.*

**Proof.** Using the results of Theorem 13, the proof of this theorem is standard.

Since $\varphi(0) \in \overline{D(A_0)}$, we have $\varphi \in \overline{D(A)}$, where $A$ is the generator of the integrated semigroup $(S(t))_{t\geq 0}$ on $\mathcal{C}_X$. Then, we deduce from Corollary 7 that $S(\cdot)\varphi$ is differentiable and $(S'(t))_{t\geq 0}$ can be defined to be a $C_0$-semigroup on $\overline{D(A)}$.

Let $(y^n)_{n\in\mathbb{N}}$ be a sequence of continuous functions defined by

$$\begin{array}{ll} y^0(t) = S'(t)\varphi, & t \in [0,T] \\ y^n(t) = S'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(s,y^{n-1}(s))\, ds, & t \in [0,T],\ n \geq 1. \end{array}$$

By virtue of the continuity of $F$ and $S'(\cdot)\varphi$, there exists $\alpha \geq 0$ such that $\left|F(s, y^0(s)\right| \leq \alpha$, for $s \in [0,T]$. Then, using Theorem 13, we obtain

$$\left|y^1(t) - y^0(t)\right| \leq M_0 \int_0^t \left|F(s, y^0(s)\right| ds.$$

Hence

$$\left|y^1(t) - y^0(t)\right| \leq M_0 \alpha t.$$

In general case we have

$$\left|y^n(t) - y^{n-1}(t)\right| \leq M_0 L \int_0^t \left|y^{n-1}(s) - y^{n-2}(s)\right| ds.$$

So,

$$\left|y^n(t) - y^{n-1}(t)\right| \leq M_0^n L^{n-1} \alpha \frac{t^n}{n!}.$$

Consequently, the limit $y := n \to \infty \lim y^n(t)$ exists uniformly on $[0,T]$ and $y : [0,T] \to \mathcal{C}_X$ is continuous.

In order to prove that $y$ is a solution of Equation (6), we introduce the function $v$ defined by

$$v(t) = \left|y(t) - S'(t)\varphi - \frac{d}{dt} \int_0^t \widetilde{S}(t-s) X_0 F(s, y(s))\, ds\right|.$$

We have

$$\begin{aligned} v(t) \leq\ & \left|y(t) - y^{n+1}(t)\right| + \left|y^{n+1}(t) - S'(t)\varphi - \tfrac{d}{dt} \int_0^t \widetilde{S}(t-s) X_0 F(s, y(s))\, ds\right|, \\ \leq\ & \left|y(t) - y^{n+1}(t)\right| + \left|\tfrac{d}{dt} \int_0^t \widetilde{S}(t-s) X_0 \left(F(s, y(s)) - F(s, y^n(s))\right) ds\right|, \\ \leq\ & \left|y(t) - y^{n+1}(t)\right| + M_0 L \int_0^t \left|y(t) - y^n(t)\right|. \end{aligned}$$

Moreover, we have

$$y(t) - y^n(t) = p = n \sum^{\infty} \left(y^{p+1}(t) - y^p(t)\right).$$

This implies that

$$v(t) \leq \quad (1 + M_0 L) \tfrac{\alpha}{L} p = n \sum^{\infty} (M_0 L)^{p+1} \tfrac{t^{p+1}}{(p+1)!} + \tfrac{\alpha}{L} (M_0 L)^{n+1} \tfrac{t^{n+1}}{(n+1)!}, \quad \text{for } n \in \mathbb{N}.$$

Consequently we obtain $v = 0$ on $[0,T]$.

To show uniqueness, suppose that $z(t)$ is also a solution of Equation (6). Then

$$|y(t) - z(t)| \leq M_0 L \int_0^t |y(s) - z(s)|\, ds.$$

By Gronwall's inequality, $z = y$ on $[0,T]$ ■

**Corollary 19** *Under the same assumptions as in Theorem 18, the solution $y : [0,T] \to \mathcal{C}_X$ of the abstract integral equation (6) is the unique integral solution of the equation*

$$\begin{cases} y'(t) = \widetilde{A}y(t) + X_0 F(t, y(t)), & t \geq 0, \\ y(0) = \varphi \in \mathcal{C}_X, \end{cases}$$

*i.e.*
**(i)** $y \in \mathcal{C}([0,T]\,; \mathcal{C}_X)$,
**(ii)** ${}_0^t y(s)ds \in D(\widetilde{A})$, *for* $t \in [0,T]$,
**(iii)** $y(t) = \varphi + \widetilde{A}{}_0^t y(s)ds + X_0{}_0^t F(s, y(s))ds$, *for* $t \in [0,T]$,
*where the operator $\widetilde{A}$ is given in Proposition 15.*
*Furthermore, we have, for $t \in [0,T]$,*

$$\|y(t)\| \leq M_0 \left( \|\varphi\| + \int_0^t |F(s, y(s))|\, ds \right).$$

**Proof.** If we define the function $f : [0,T] \to \widetilde{\mathcal{C}}_X$ by $f(s) = X_0 F(s, y(s))$, we can use Theorem 13 and Theorem 18 to prove this result ■

**Corollary 20** *Under the same assumptions as in Theorem 18, the solution $y$ of the integral equation (6) satisfies, for $t \in [0,T]$ and $\theta \in [-r, 0]$ the translation property*

$$y(t)(\theta) = \begin{cases} y(t+\theta)(0) & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases}$$

*Moreover, if we consider the function $u : [-r, T] \to X$ defined by*

$$u(t) = \begin{cases} y(t)(0) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t \leq 0. \end{cases}$$

*Then, $u$ is the unique integral solution of the problem (1), i.e.*

$$\begin{cases} \textbf{(i)}\ u \in \mathcal{C}([-r,T]\,; X), & \\ \textbf{(ii)}\ {}_0^t u(s)ds \in D(A_0), & \text{for } t \in [0,T], \\ \textbf{(iii)}\ u(t) = \begin{cases} \varphi(0) + A_0{}_0^t u(s)ds +_0^t F(s, u_s)ds, & \text{for } t \in [0,T], \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases} \end{cases} \tag{7}$$

*The function $u \in \mathcal{C}([-r,T]\,; X)$ is also the unique solution of*

$$u(t) = \begin{cases} S_0'(t)\varphi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S_0(t-s)F(s, u_s)ds, & \text{for } t \in [0,T], \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases} \tag{8}$$

*Furthermore, we have for $t \in [0,T]$*

$$\|u_t\| \leq M_0 \left( \|\varphi\| + \int_0^t |F(s,u_s)|\, ds \right).$$

*Conversely, if $u$ is an integral solution of Equation (1), then the function $t \mapsto u_t$ is a solution of Equation (6).*

**Proof.** From Proposition 14 and Proposition 15, we have for $t+\theta \geq 0$

$$\begin{aligned} y(t)(\theta) = &\quad S_0'(t+\theta)\varphi(0) + \frac{d}{dt}\int_0^{t+\theta} S_0(t+\theta-s)F(s,y(s))\, ds, \\ = &\quad y(t+\theta)\varphi(0), \end{aligned}$$

and for $t+\theta \leq 0$, we obtain $\int_0^t \left(\widetilde{S}(t-s)X_0F(s,y(s))\right)(\theta)\, ds = 0$ and $\left(S'(t)\varphi\right)(\theta) = \varphi(t+\theta)$. Hence

$$y(t)(\theta) = \varphi(t+\theta).$$

The second part of the corollary follows from Corollary 19 ■

**Remark 2** *A continuous function $u$ from $[-r,T]$ into $X$ is called an integral solution of Equation (1) if the function $t \mapsto u_t$ satisfies Equation (6) or equivalently $u$ satisfies (7) or equivalently $u$ satisfies (8).*

**Corollary 21** *Assume that the hypotheses of Theorem 18 are satisfied and let $u$, $\widehat{u}$ be the functions given by Corollary 20 for $\varphi, \widehat{\varphi} \in \mathcal{C}_X$, respectively. Then, for $t \in [0,T]$*

$$\|u_t - \widehat{u}_t\| \leq M_0 e^{Lt} \|\varphi - \widehat{\varphi}\|.$$

**Proof.** This is just a consequence of the last inequality stated in Corollary 20. After the equation (6) has been centered near $u$, we obtain the equation

$$v(t) = S'(t)\left(\widehat{\varphi} - \varphi\right) + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0\left(F(s,u_s+v(s)) - F\left(s,u_s\right)\right) ds,$$

with

$$v(t) = \widehat{u}_t - u_t.$$

This yields

$$\|v(t)\| \leq M_0 \left( \|\varphi - \widehat{\varphi}\| + \int_0^t |F(s,u_s) - F(s,\widehat{u}_s)|\, ds \right),$$

this is

$$\|u_t - \widehat{u}_t\| \leq M_0 \left( \|\varphi - \widehat{\varphi}\| + L \int_0^t \|u_s - \widehat{u}_s\| \, ds \right).$$

By Gronwall's inequality, we obtain

$$\|u_t - \widehat{u}_t\| \leq M_0 e^{Lt} \|\varphi - \widehat{\varphi}\| \quad \blacksquare\blacksquare$$

We give now two results of regularity of the integral solutions of (1).

**Theorem 22** *Assume that $F : [0,T] \times \mathcal{C}_X \to X$ is continuously differentiable and there exist constants $L, \beta, \gamma \geq 0$ such that*

$$\begin{array}{l} |F(t,\varphi) - F(t,\psi)| \leq L \|\varphi - \psi\|, \\ |D_t F(t,\varphi) - D_t F(t,\psi)| \leq \beta \|\varphi - \psi\|, \\ |D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)| \leq \gamma \|\varphi - \psi\|, \end{array}$$

*for all $t \in [0,T]$ and $\varphi, \psi \in \mathcal{C}_X$, where $D_t F$ and $D_\varphi F$ denote the derivatives. Then, for given $\varphi \in \mathcal{C}_X$ such that*

$$\varphi(0) \in D(A_0), \;\; \varphi' \in \mathcal{C}_X, \;\; \varphi'(0) \in \overline{D(A_0)} \text{ and } \varphi'(0) = A_0\varphi(0) + F(0,\varphi),$$

*let $y : [0,T] \to \mathcal{C}_X$ be the solution of the abstract integral equation (6) such that $y(0) = \varphi$. Then, $y$ is continuously differentiable on $[0,T]$ and satisfies the Cauchy problem*

$$\begin{cases} y'(t) = \widetilde{A}y(t) + X_0 F(t, y(t)), & t \in [0,T], \\ y(0) = \varphi. \end{cases}$$

*Moreover, the function $u$ defined on $[-r,T]$ by*

$$u(t) = \begin{cases} y(t)(0) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t < 0, \end{cases}$$

*is continuously differentiable on $[-r,T]$ and satisfies the Cauchy problem (1).*

**Proof.** Let $y$ be the solution of Equation (6) on $[0,T]$ such that $y(0) = \varphi$. We deduce from Theorem 18 that there exists a unique function $v : [0,T] \to \mathcal{C}_X$ which solves the following integral equation

$$v(t) = S'(t)\varphi' + \frac{d}{dt}\int_0^t \widetilde{S}(t-s) X_0 \left( D_t F(s,y(s)) + D_\varphi F(s,y(s)) v(s) \right) ds,$$

such that $v(0) = \varphi'$.
Let $w : [0,T] \to \mathcal{C}_X$ be the function defined by

$$w(t) = \varphi + \int_0^t v(s)\, ds, \ \text{ for } t \in [0,T]\,.$$

We will show that $w = y$ on $[0,T]$.

Using the expression satisfied by $v$, we obtain

$$\begin{aligned} w(t) = &\ \ \varphi + \int_0^t S'(s)\varphi'\, ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,y(s)) + D_\varphi F(s,y(s))v(s)\right)\, ds, \\ \\ = &\ \ \varphi + S(t)\varphi' + \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,y(s)) + D_\varphi F(s,y(s))v(s)\right)\, ds. \end{aligned}$$

On the other hand, we have $\varphi \in D(\widetilde{A})$ and $\varphi'(0) = A_0\varphi(0) + F(0,\varphi)$, then $\varphi' = \widetilde{A}\varphi + X_0F(0,\varphi)$. This implies that

$$S(t)\varphi' = \widetilde{S}(t)\varphi' = \widetilde{S}(t)\widetilde{A}\varphi + \widetilde{S}(t)X_0F(0,\varphi).$$

Using Corollary 7, we deduce that

$$S(t)\varphi' = S'(t)\varphi - \varphi + \widetilde{S}(t)X_0F(0,\varphi).$$

Furthermore, we have

$$\frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(s,w(s))\, ds = \frac{d}{dt}\int_0^t \widetilde{S}(s)X_0F(s,w(t-s))\, ds$$

$$= \int_0^t \widetilde{S}(t-s)X_0\left[D_tF(s,w(s)) \ + D_\varphi F(s,w(s))\right]v(s)ds + \widetilde{S}(t)X_0F(0,\varphi).$$

Then

$$\begin{aligned} w(t) = &\ \ S'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(s,w(s))ds \\ &- \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,w(s)) + D_\varphi F(s,w(s))v(s)\right)\, ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,y(s)) + D_\varphi F(s,y(s))v(s)\right)\, ds, \\ \\ = &\ \ S'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(s,w(s))ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,y(s)) - D_tF(s,w(s))\right)\, ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_\varphi F(s,y(s)) - D_\varphi F(s,w(s))\right)v(s)\, ds. \end{aligned}$$

We obtain

$$\begin{aligned} w(t) - y(t) = &\ \ \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0\left(F(s,w(s)) - F(s,y(s))\right) ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_tF(s,y(s)) - D_tF(s,w(s))\right)\, ds \\ &+ \int_0^t \widetilde{S}(t-s)X_0\left(D_\varphi F(s,y(s)) - D_\varphi F(s,w(s))\right)v(s)\, ds. \end{aligned}$$

So, we deduce

$$|w(t) - y(t)| \leq M_0 \int_0^t (L + \beta + \gamma |v(s)|) |w(s) - y(s)| \, ds.$$

By Gronwall's inequality, we conclude that $w = y$ on $[0, T]$. This implies that $y$ is continuously differentiable on $[0, T]$.

Consider now the function $g : [0, T] \to \widetilde{\mathcal{C}}_X$ defined by $g(t) = X_0 F(t, y(t))$ and consider the Cauchy problem

$$\begin{cases} z'(t) = \widetilde{A} z(t) + g(t), & t \in [0, T], \\ z(0) = \varphi. \end{cases} \tag{9}$$

The assumptions of Theorem 22 imply that $\varphi \in D(\widetilde{A})$, $\widetilde{A}\varphi + g(0) \in \overline{D(\widetilde{A})}$ and $g$ is continuously differentiable on $[0, T]$. Using Theorem 11, we deduce that there exists a unique solution on $[0, T]$ of Equation (9). By Theorem 13, we know that this solution is given by

$$z(t) = S'(t)\varphi + \frac{d}{dt} \int_0^t \widetilde{S}(t - s) g(s) ds.$$

Theorem 18 implies that $z = y$ on $[0, T]$.

If we consider the function $u$ defined on $[-r, T]$ by

$$u(t) = \begin{cases} y(t)(0) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t < 0. \end{cases}$$

By virtue of Corollary 20, we have $\int_0^t u(s) ds \in D(A_0)$ and

$$u(t) = \varphi(0) + A_0 \int_0^t u(s) ds + \int_0^t F(s, u_s) ds, \qquad \text{for } t \in [0, T].$$

We have also the existence of

$$h \to 0 \lim A_0 \left( \frac{1}{h} \int_t^{t+h} u(s) \, ds \right) = u'(t) - F(t, u_t).$$

Furthermore, the operator $A_0$ is closed. Then, we obtain $u(t) \in D(A_0)$ and

$$u'(t) = A_0 u(t) + F(t, u_t), \ \text{ for } t \in [0, T].$$

The second part of the theorem is a consequence of Corollary 19 ■

Assume that $T > r$ and $A_0 : D(A_0) \subseteq X \to X$ satisfies (with not necessarily dense domain) the condition

$$\begin{cases} \text{there exist } \beta \in \left]\frac{\pi}{2}, \pi\right[ \text{ and } M_0 > 0 \text{ such that if} \\ \lambda \in \mathbb{C} - \{0\} \text{ and } |\arg \lambda| < \beta, \text{ then} \\ \left\|(\lambda I - A_0)^{-1}\right\| \leq \frac{M_0}{|\lambda|}. \end{cases} \tag{10}$$

The condition (10) is stronger than (**HY**).

We have the following result.

**Theorem 23** *Suppose that $A_0$ satisfies (10) (non-densely defined) on $X$ and there exist a constant $L > 0$ and $\alpha \in ]0, 1[$ such that*

$$|F(t, \psi) - F(s, \varphi)| \leq L \left(|t - s|^\alpha + \|\psi - \varphi\|\right)$$

*for $t, s \in [0, T]$ and $\psi, \varphi \in \mathcal{C}_X$.*
*Then, for given $\varphi \in \mathcal{C}_X$, such that $\varphi(0) \in \overline{D(A_0)}$, the integral solution $u$ of Equation (1) on $[0, T]$ is continuously differentiable on $(r, T]$ and satisfies*

$$\begin{array}{l} u(t) \in D(A_0),\ u'(t) \in \overline{D(A_0)} \text{ and} \\ u'(t) = A_0 u(t) + F(t, u_t), \qquad \text{for } t \in (r, T]. \end{array}$$

**Proof.** We know, from ([48], p.487) that $A_0$ is the generator of an analytic semigroup (not necessarily $C_0-$semigroup) defined by

$$e^{A_0 t} = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} (\lambda I - A_0)^{-1} d\lambda, \quad t > 0$$

where $+C$ is a suitable oriented path in the complex plan.

Let $u$ be the integral solution on $[0, T]$ of Equation (1), which exists by virtu of Theorem 18, and consider the function $g : [0, T] \to X$ defined by $g(t) = F(t, u_t)$. We deduce from ([60], p.106) that

$$u(t) = e^{A_0 t}\varphi(0) + \int_0^t e^{A_0(t-s)} g(s)\, ds, \text{ for } t \in [0, T].$$

Using (Theorem 3.4, [62]), we obtain that the function $u$ is $\gamma-$Hölder continuous on $[\varepsilon, T]$ for each $\varepsilon > 0$ and $\gamma \in ]0, 1[$. Hence, there exists $L_1 \geq 0$ such that

$$\|u_t - u_s\| \leq L_1 |t - s|^\gamma, \quad \text{for } t, s \in (r, T].$$

On the other hand, we have

$$|g(t) - g(s)| \leq L(|t-s|^{\alpha} + \|u_t - u_s\|), \quad \text{for } t, s \in [0, T] .$$

Consequently, the function $t \in (r, T] \to g(t)$ is locally Hölder continuous.

By virtue of (Theorem 4.4 and 4.5, [62]), we deduce that $u$ is continuously differentiable on $(r, T]$ and satisfies

$$\begin{array}{l} u(t) \in D(A_0), \;\; u'(t) \in \overline{D(A_0)} \text{ and} \\ u'(t) = A_0 u(t) + F(t, u_t), \qquad \text{for } t \in (r, T] \quad \blacksquare\blacksquare \end{array}$$

We prove now the local existence of integral solutions of Problem (1) under a locally Lipschitz condition on $F$.

**Theorem 24** *Suppose that $F : [0, +\infty) \times \mathcal{C}_X \to X$ is continuous and satisfies the following locally Lipschitz condition : for each $\alpha > 0$ there exists a constant $C_0(\alpha) > 0$ such that if $t \geq 0$, $\varphi_1, \varphi_2 \in \mathcal{C}_X$ and $\|\varphi_1\|, \|\varphi_2\| \leq \alpha$ then*

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq C_0(\alpha) \|\varphi_1 - \varphi_2\| .$$

*Let $\varphi \in \mathcal{C}_X$ such that $\varphi(0) \in \overline{D(A_0)}$. Then, there exists a maximal interval of existence $[-r, T_\varphi[$ , $T_\varphi > 0$, and a unique integral solution $u(., \varphi)$ of Equation (1), defined on $[-r, T_\varphi[$ and either*

$$T_\varphi = +\infty \qquad \textit{or} \qquad t \to T_\varphi^- \limsup |u(t, \varphi)| = +\infty.$$

*Moreover, $u(t, \varphi)$ is a continuous function of $\varphi$, in the sense that if $\varphi \in \mathcal{C}_X$, $\varphi(0) \in \overline{D(A_0)}$ and $t \in [0, T_\varphi[$ , then there exist positive constants $L$ and $\varepsilon$ such that, for $\psi \in \mathcal{C}_X$, $\psi(0) \in \overline{D(A_0)}$ and $\|\varphi - \psi\| < \varepsilon$, we have*

$$t \in [0, T_\psi[ \;\; \textit{and} \;\; |u(s, \varphi) - u(s, \psi)| \leq L \|\varphi - \psi\| , \; \textit{for all } s \in [-r, t] .$$

**Proof.** Let $T_1 > 0$. Note that the locally Lipschitz condition on $F$ implies that, for each $\alpha > 0$ there exists $C_0(\alpha) > 0$ such that for $\varphi \in \mathcal{C}_X$ and $\|\varphi\| \leq \alpha$, we have

$$|F(t, \varphi)| \leq C_0(\alpha) \|\varphi\| + |F(t, 0)| \leq \alpha C_0(\alpha) + s \in [0, T_1] \sup |F(s, 0)| .$$

Let $\varphi \in \mathcal{C}_X$, $\varphi(0) \in \overline{D(A_0)}$, $\alpha = \|\varphi\| + 1$ and $c_1 = \alpha C_0(\alpha) + s \in [0, T_1] \sup |F(s, 0)|$ . Consider the following set

$$\begin{array}{rl} Z_\varphi = \{ u \in \mathcal{C}([-r, T_1], X) : u(s) = \varphi(s) & \text{if } s \in [-r, 0] \\ & \text{and } 0 \leq s \leq T_1 \sup |u(s) - \varphi(0)| \leq 1 \} , \end{array}$$

where $\mathcal{C}([-r,T_1],X)$ is endowed with the uniform convergence topology. It's clear that $Z_\varphi$ is a closed set of $\mathcal{C}([-r,T_1],X)$. Consider the mapping

$$H : Z_\varphi \to \mathcal{C}([-r,T_1],X)$$

defined by

$$H(u)(t) = \begin{cases} S_0'(t)\varphi(0) + \dfrac{d}{dt}{}_0^t S_0(t-s)F(s,u_s)ds, & \text{for } t \in [0,T_1], \\ \varphi(t), & \text{for } t \in [-r,0]. \end{cases}$$

We will show that

$$H(Z_\varphi) \subseteq Z_\varphi.$$

Let $u \in Z_\varphi$ and $t \in [0,T_1]$, we have, for suitable constants $M_0$ and $\omega_0$

$$\begin{aligned} |H(u)(t) - \varphi(0)| &\le |S_0'(t)\varphi(0) - \varphi(0)| + \left| \frac{d}{dt}{}_0^t S_0(t-s)F(s,u_s)ds \right|, \\ &\le |S_0'(t)\varphi(0) - \varphi(0)| + M_0 e_0^{\omega_0 t t} e^{-\omega_0 s} |F(s,u_s)| ds. \end{aligned}$$

We can assume, without loss of generality, that $\omega_0 > 0$. Then,

$$|H(u)(t) - \varphi(0)| \le \left|S_0'(t)\varphi(0) - \varphi(0)\right| + M_0 e_0^{\omega_0 t t} |F(s,u_s)| ds.$$

Since $|u(s) - \varphi(0)| \le 1$, for $s \in [0,T_1]$, and $\alpha = \|\varphi\| + 1$, we obtain $|u(s)| \le \alpha$, for $s \in [-r,T_1]$. Then, $\|u_s\| \le \alpha$, for $s \in [0,T_1]$ and

$$\begin{aligned} |F(s,u_s)| &\le C_0(\alpha)\|u_s\| + |F(s,0)|, \\ &\le c_1. \end{aligned}$$

Consider $T_1 > 0$ sufficiently small such that

$$0 \le s \le T_1 \sup\left\{ \left|S_0'(s)\varphi(0) - \varphi(0)\right| + M_0 e^{\omega_0 s} c_1 s \right\} < 1.$$

So, we deduce that

$$\begin{aligned} |H(u)(t) - \varphi(0)| &\le |S_0'(t)\varphi(0) - \varphi(0)| + M_0 e^{\omega_0 t} c_1 t \\ &< 1, \end{aligned}$$

for $t \in [0,T_1]$. Hence,

$$H(Z_\varphi) \subseteq Z_\varphi.$$

On the other hand, let $u, v \in Z_\varphi$ and $t \in [0, T_1]$, we have

$$\begin{aligned} |H\,(u)\,(t) - H\,(v)\,(t)| &= \left| \frac{d}{dt}{}_0^{t} S_0(t-s)(F(s,u_s) - F(s,v_s))ds \right|, \\ &\leq M_0 e_0^{\omega_0 t t} \left| F(s,u_s) - F(s,v_s) \right| ds, \\ &\leq M_0 e^{\omega_0 t} C_0\,(\alpha)_0^t \left\| u_s - v_s \right\| ds, \\ &\leq M_0 e^{\omega_0 T_1} C_0\,(\alpha)\, T_1 \left\| u - v \right\|_{\mathcal{C}([-r,T_1],X)}. \end{aligned}$$

Note that $\alpha \geq 1$, then

$$\begin{aligned} M_0 e^{\omega_0 T_1} C_0(\alpha) T_1 &\leq M_0 e^{\omega_0 T_1} c_1 T_1, \\ &\leq 0 \leq s \leq T_1 \sup \left\{ |S_0'(s)\varphi(0) - \varphi(0)| + M_0 e^{\omega_0 s} c_1 s \right\}, \\ &< 1. \end{aligned}$$

Then, $H$ is a strict contraction in $Z_\varphi$. So, $H$ has one and only one fixed point $u$ in $Z_\varphi$. We conclude that Equation (1) has one and only one integral solution which is defined on the interval $[-r, T_1]$.

Let $u(.,\varphi)$ be the unique integral solution of Equation (1), defined on it's maximal interval of existence $[0, T_\varphi[\,,\ T_\varphi > 0$.

Assume that $T_\varphi < +\infty$ and $t \to T_\varphi^- \limsup |u(t,\varphi)| < +\infty$. Then, there exists a constant $\alpha > 0$ such that $\|u(t,\varphi)\| \leq \alpha$, for $t \in [-r, T_\varphi[$. Let $t, t+h \in [0, T_\varphi[\,,\ h > 0$, and $\theta \in [-r, 0]$.
If $t + \theta \geq 0$, we obtain

$$\begin{aligned} &|u(t+\theta+h,\varphi) - u(t+\theta,\varphi)| \leq |(S_0'(t+\theta+h) - S_0'(t+\theta))\varphi(0)| \\ &+ \left| \frac{d}{dt}{}_0^{t+\theta+h} S_0(t+\theta+h-s)F(s,u_s(.,\varphi))ds - \frac{d}{dt}{}_0^{t+\theta} S_0(t+\theta-s)F(s,u_s(.,\varphi))ds \right|, \\ &\leq \|S_0'(t+\theta)\| \, |S_0'(h)\varphi(0) - \varphi(0)| + \left| \frac{d}{dt}{}_{t+\theta}^{t+\theta+h} S_0(s)F(t+\theta+h-s, u_{t+\theta+h-s}(.,\varphi))ds \right| \\ &\quad + \left| \frac{d}{dt}{}_0^{t+\theta} S_0(s) \left( F(u_{t+\theta+h-s},\varphi) - F(t+\theta-s, u_{t+\theta-s}(.,\varphi)) \right) ds \right|. \end{aligned}$$

This implies that,

$$\begin{aligned} |u_{t+h}(\theta,\varphi) - u_t(\theta,\varphi)| &\leq M_0 e^{\omega_0 T_\varphi} |S_0'(h)\varphi(0) - \varphi(0)| + M_0 e^{\omega_0 T_\varphi} c_1 h \\ &\quad + M_0 e^{\omega_0 T_\varphi} C_0(\alpha)_0^t \|u_{s+h}(.,\varphi) - u_s(.,\varphi)\| \, ds. \end{aligned}$$

If $t + \theta < 0$. Let $h_0 > 0$ sufficiently small such that, for $h \in \,]0, h_0[$

$$|u_{t+h}(\theta,\varphi) - u_t(\theta,\varphi)| \leq -r \leq \sigma \leq 0 \sup |u(\sigma+h,\varphi) - u(\sigma,\varphi)|.$$

Consequently, for $t, t+h \in [0, T_\varphi[\,,\ h \in ]0, h_0[\,;$

$$\begin{aligned} \|u_{t+h}(.,\varphi) - u_t(,\varphi)\| \quad & \leq \delta(h) + M_0 e^{\omega_0 T_\varphi} \left( |S_0'(h)\varphi(0) - \varphi(0)| + c_1 h \right) \\ & +M_0 e^{\omega_0 T_\varphi} C_0(\alpha)_0^t \, \|u_{s+h}(.,\varphi) - u_s(.,\varphi)\| \, ds, \end{aligned}$$

where

$$\delta(h) = -r \leq \sigma \leq 0 \mathrm{sup} \left| u(\sigma+h,\varphi) - u(\sigma,\varphi) \right| .$$

By Gronwall's Lemma, it follows

$$\|u_{t+h}(.,\varphi) - u_t(,\varphi)\| \leq \beta(h) \exp\left[ C_0(\alpha) M_0 e^{\omega_0 T_\varphi} T_\varphi \right],$$

with

$$\beta(h) = \delta(h) + M_0 e^{\omega_0 T_\varphi} \left[ \left| S_0'(h)\varphi(0) - \varphi(0) \right| + c_1 h \right].$$

Using the same reasoning, one can show the same result for $h < 0$.
It follows immediately, that

$$t \to T_\varphi^- \mathrm{lim} u(t,\varphi) \ \text{ exists.}$$

Consequently, $u(.,\varphi)$ can be extended to $T_\varphi$, which contradicts the maximality of $[0, T_\varphi[\,.$

We will prove now that the solution depends continuously on the initial data. Let $\varphi \in \mathcal{C}_X$, $\varphi(0) \in \overline{D(A_0)}$ and $t \in [0, T_\varphi[$. We put

$$\alpha = 1 + -r \leq s \leq t \mathrm{sup} \left| u(s,\varphi) \right|$$

and

$$c(t) = M_0 e^{\omega_0 t} \exp\left( M_0 e^{\omega_0 t} C_0(\alpha) t \right).$$

Let $\varepsilon \in ]0,1[$ such that $c(t)\varepsilon < 1$ and $\psi \in \mathcal{C}_X$, $\psi(0) \in \overline{D(A_0)}$ such that

$$\|\varphi - \psi\| < \varepsilon.$$

We have

$$\|\psi\| \quad \leq \|\varphi\| + \varepsilon < \alpha.$$

Let

$$T_0 = \sup\left\{ s > 0 : \|u_\sigma(.,\psi)\| \leq \alpha \text{ for } \sigma \in [0,s] \right\}.$$

If we suppose that $T_0 < t$, we obtain for $s \in [0, T_0]\,,$

$$\|u_s(.,\varphi) - u_s(.,\psi)\| \leq M_0 e^{\omega_0 t} \|\varphi - \psi\| + M_0 e^{\omega_0 t} C_0(\alpha) \int_0^s \|u_\sigma(.,\varphi) - u_\sigma(.,\psi)\| \, d\sigma.$$

By Gronwall's Lemma, we deduce that

$$\|u_s(,\varphi) - u_s(,\psi)\| \leq c(t)\,\|\varphi - \psi\|\,. \tag{11}$$

This implies that

$$\|u_s(.,\psi)\| \leq c(t)\varepsilon + \alpha - 1 < \alpha, \text{ for all } s \in [0, T_0]\,.$$

It follows that $T_0$ cannot be the largest number $s > 0$ such that $\|u_\sigma(.,\psi)\| \leq \alpha$, for $\sigma \in [0,s]$. Thus, $T_0 \geq t$ and $t < T_\psi$. Furthermore, $\|u_s(.,\psi)\| \leq \alpha$, for $s \in [0,t]$, then using the inequality (11) we deduce the dependence continuous with the initial data. This completes the proof of Theorem ■

**Theorem 25** *Assume that $F$ is continuously differentiable and satisfies the following locally Lipschitz condition: for each $\alpha > 0$ there exists a constant $C_1(\alpha) > 0$ such that if $\varphi_1, \varphi_2 \in \mathcal{C}_X$ and $\|\varphi_1\|, \|\varphi_2\| \leq \alpha$ then*

$$\begin{aligned} &|F(t,\varphi_1) - F(t,\varphi_2)| \leq C_1(\alpha)\,\|\varphi_1 - \varphi_2\|\,,\\ &|D_tF(t,\varphi_1) - D_tF(t,\varphi_2)| \leq C_1(\alpha)\,\|\varphi_1 - \varphi_2\|\,,\\ &|D_\varphi F(t,\varphi_1) - D_\varphi F(t,\varphi_2)| \leq C_1(\alpha)\,\|\varphi_1 - \varphi_2\|\,. \end{aligned}$$

*Then, for given $\varphi \in \mathcal{C}_X^1 := \mathcal{C}^1([-r,0], X)$ such that*

$$\varphi(0) \in D(A_0), \;\; \varphi'(0) \in \overline{D(A_0)} \text{ and } \varphi'(0) = A\varphi(0) + F(\varphi),$$

*let $u(.,\varphi) : [-r, T_\varphi[ \;\to X$ be the unique integral solution of Equation (1). Then, $u(.,\varphi)$ is continuously differentiable on $[-r, T_\varphi[$ and satisfies Equation (1).*

**Proof.** The proof is similar to the proof of Theorem 22 ■

# 4 The semigroup and the integrated semigroup in the autonomous case

Let $E$ be defined by

$$E := \overline{D(A)}^{\mathcal{C}_X} = \left\{\varphi \in \mathcal{C}_X : \;\; \varphi(0) \in \overline{D(A_0)}\right\}.$$

Throughout this section we will suppose the hypothesis of Theorem 18 except that we require $F$ to be autonomous, that is, $F : \mathcal{C}_X \to X$. By virtue of Theorem 18, there exists for each $\varphi \in E$ a unique continuous function $y(.,\varphi) : [0,+\infty) \to X$ satisfying the following equation

$$y(t) = S'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{S}(t-s)X_0F(y(s))\,ds, \qquad \text{for } t \geq 0. \tag{12}$$

Let us consider the operator $T(t) : E \to E$ defined, for $t \geq 0$ and $\varphi \in E$, by

$$T(t)(\varphi) = y(t,\varphi).$$

Then, we have the following result.

**Proposition 26** *Under the same assumptions as in Theorem 18, the family $(T(t))_{t\geq 0}$ is a nonlinear strongly continuous semigroup of continuous operators on $E$, that is*
**i**) $T(0) = Id$,
**ii**) $T(t+s) = T(t)T(s)$, *for all* $t, s \geq 0$,
**iii**) *for all* $\varphi \in E$, $T(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* $E$,
**iv**) *for all* $t \geq 0$, $T(t)$ *is continuous from* $E$ *into* $E$.
*Moreover,*
**v**) $(T(t))_{t\geq 0}$ *satisfies, for* $t \geq 0$, $\theta \in [-r,0]$ *and* $\varphi \in E$, *the translation property*

$$(T(t)(\varphi))(\theta) = \begin{cases} (T(t+\theta)(\varphi))(0), & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0, \end{cases} \tag{13}$$

*and*
**vi**) *there exists* $\gamma > 0$ *and* $M \geq 0$ *such that,*

$$\|T(t)(\varphi_1) - T(t)(\varphi_2)\| \leq Me^{\gamma t}\|\varphi_1 - \varphi_2\|, \text{ for all } \varphi_1, \varphi_2 \in E.$$

**Proof.** The proof of this proposition is standard.
We recall the following definition.

**Definition 27** *A $C_0-$semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, |\cdot|)$ is called compact if all operators $T(t)$ are compact on $X$ for $t > 0$.*

**Lemma 28** *Let $(S(t))_{t\geq 0}$ a locally Lipschitz continuous integrated semigroup on a Banach space $(X, |\cdot|)$, $(A, D(A))$ its generator, $B$ a bounded subset of $X$, $\{G_\lambda,\ \lambda \in \Lambda\}$ a set of continuous functions from the finite interval $[0,T]$ into $B$, $c > 0$ a constant and*

$$P_\lambda(t) = \frac{d}{dt}\int_0^{t-c} S(t-s)G_\lambda(s)\,ds, \qquad \text{for } c < t \leq T + c \text{ and } \lambda \in \Lambda.$$

*Assume that $S'(t) : (\overline{D(A)}, |\cdot|) \to (\overline{D(A)}, |\cdot|)$ is compact for each $t > 0$. Then, for each $t \in (c, T+c]$, $\{P_\lambda(t),\ \lambda \in \Lambda\}$ is a precompact subset of $(\overline{D(A)}, |\cdot|)$.*

**Proof.** The proof is similar to the proof of a fundamental result of Travis and Webb [65]. Let $H = \left\{S'(t)x;\ \ t \in [c, T+c]\,,\ \ x \in \overline{D(A)},\ |x| \leq kN\right\}$, where $k$ is the Lipschitz constant of $S(\cdot)$ on $[0, T]$ and $N$ is a bound of $B$. Using the same reasoning as in the proof of Lemma 2.5 [65], one can prove that $H$ is precompact in $(\overline{D(A)}, |\cdot|)$. Hence, the convex hull of $H$ is precompact. On the other hand, we have, from Lemma 17

$$P_\lambda(t) = h \searrow 0 \lim \frac{1}{h} \int_0^{t-c} S'(t-s)S(h)G_\lambda(s)\, ds,$$

and, for each $h > 0$ small enough and $t \in (c, T+c]$ fixed, the set

$$\left\{S'(s)\frac{1}{h}S(h)G_\lambda(t-s),\ \ s \in [c, t]\,,\ \lambda \in \Lambda\right\}$$

is contained in $H$. Then, the set

$$\left\{\frac{1}{h}\int_c^t S'(s)S(h)G_\lambda(t-s)\, ds,\ \lambda \in \Lambda\right\}$$

is contained in the closed convex hull of $(t-c)H$. Letting $h$ tend to zero, the set

$$\{P_\lambda(t),\ \ \lambda \in \Lambda\}$$

is still a precompact set of $\overline{D(A)}$. Hence the proof is complete ■

**Theorem 29** *Assume that $S_0'(t) : (\overline{D(A_0)}, |\cdot|) \to (\overline{D(A_0)}, |\cdot|)$ is compact for each $t > 0$ and the assumptions of Theorem 18 are satisfied. Then, the nonlinear semigroup $(T(t))_{t\geq 0}$ is compact on $(E, \|.\|_{\mathcal{C}_X})$ for every $t > r$.*

**Proof.** Let $\{\varphi_\lambda, \lambda \in \Lambda\}$ be a bounded subset of $E$ and let $t > r$. For $\theta \in [-r, 0]$, we have $t + \theta > 0$. For each $\lambda \in \Lambda$, define $y_\lambda$ by

$$y_\lambda(t)(\theta) = y(t, \varphi_\lambda)(\theta).$$

Then, we obtain

$$\begin{aligned} y_\lambda(t)(\theta) = {} & S_0'(t+\theta)\varphi_\lambda(0) + \tfrac{d}{dt}\textstyle\int_0^{t+\theta} S_0(t+\theta-s)F(y_\lambda(s))\, ds, \\ = {} & S_0'(t+\theta)\varphi_\lambda(0) + h \searrow 0 \lim \tfrac{1}{h}\textstyle\int_0^{t+\theta} S_0'(t+\theta-s)S_0(h)F(y_\lambda(s))\, ds. \end{aligned}$$

First, we show that the family $\{y_\lambda(t),\ \lambda \in \Lambda\}$ is equicontinuous, for each $t > r$. Using the Lipschitz condition on $F$, one shows that $\{F(y_\lambda(s)),\ \ s \in [0,t] \ \text{ and } \lambda \in \Lambda\}$ is bounded by a constant, say $K$.

Let $\lambda \in \Lambda$, $0 < c < t - r$ and $-r \leq \widehat{\theta} < \theta \leq 0$. Observe that

$$\begin{aligned}
\left| y_\lambda(t)(\theta) - y_\lambda(t)(\widehat{\theta})\right| \leq & \left| S_0'(t+\theta)\varphi_\lambda(0) - S_0'(t+\widehat{\theta})\varphi_\lambda(0)\right| \\
& + h \searrow 0 \lim \tfrac{1}{h} \left| \int_{t+\widehat{\theta}}^{t+\theta} S_0'(t+\theta-s) S_0(h) F(y_\lambda(s))\, ds \right| \\
& + h \searrow 0 \lim \tfrac{1}{h} \left| \int_{t+\widehat{\theta}-c}^{t+\widehat{\theta}} \left( S_0'(t+\theta-s) - S_0'(t+\widehat{\theta}-s)\right) S_0(h) F(y_\lambda(s))\, ds \right| \\
& + h \searrow 0 \lim \tfrac{1}{h} \left| \int_{0}^{t+\widehat{\theta}-c} \left( S_0'(t+\theta-s) - S_0'(t+\widehat{\theta}-s)\right) S_0(h) F(y_\lambda(s))\, ds \right|,
\end{aligned}$$

$$\begin{aligned}
\leq & \left\| S_0'(t+\theta) - S_0'(t+\widehat{\theta})\right\| |\varphi_\lambda(0)| + M_0 e^{\omega_0 t} kK \left| \theta - \widehat{\theta}\right| + c2M_0 e^{\omega_0 t} kK \\
& + ts \in \left[0, t+\widehat{\theta} - c\right] \sup \left\| S_0'(t+\theta-s) - S_0'(t+\widehat{\theta}-s)\right\| kK.
\end{aligned}$$

If we take $\left|\theta - \widehat{\theta}\right|$ small enough, it follows from the uniform continuity of $S_0'(.) : [a,t] \to \mathcal{L}(\overline{D(A_0)})$ for $0 < a < t$, the claimed equicontinuity of $\{y_\lambda(t),\ \lambda \in \Lambda\}$.

On the other hand, the family $\{S_0'(t+\theta)\varphi_\lambda(0);\ \lambda \in \Lambda\}$ is precompact, for each $t > r$ and $\theta \in [-r, 0]$. We will show that, for each $t > r$ and $\theta \in [-r, 0]$.

$$\left\{ \frac{d}{dt} \int_0^{t+\theta} S_0(t+\theta-s) F(y_\lambda(s))\, ds;\ \lambda \in \Lambda \right\}$$

is precompact.

Observe that for $0 < c < t + \theta$ and $\lambda \in \Lambda$

$$\frac{1}{h} \left| \int_{t+\theta-c}^{t+\theta} S_0'(t+\theta-s) S_0(h) F(y_\lambda(s))\, ds \right| \leq c M_0 e^{\omega_0 t} kK.$$

Then, if $h$ tends to zero we obtain

$$\left| \frac{d}{dt} \int_{t+\theta-c}^{t+\theta} S_0(t+\theta-s) F(y_\lambda(s))\, ds \right| \leq c M_0 e^{\omega_0 t} kK. \tag{14}$$

For $0 < c < t + \theta$ and $\lambda \in \Lambda$,

$$\int_0^{t+\theta-c} S_0(t+\theta-s) F(y_\lambda(s))\, ds = \int_c^{t+\theta} S_0(s) F(y_\lambda(t+\theta-s))\, ds.$$

By lemma 28, if $0 < c < t + \theta$ and $\lambda \in \Lambda$, then

$$\left\{ \frac{d}{dt} \int_0^{t+\theta-c} S_0(t+\theta-s)F(y_\lambda(s))\, ds, \quad \lambda \in \Lambda \right\}$$

is precompact in $X$, for each $t > r$ and $\theta \in [-r, 0]$. This fact together with (14) yields the precompactness of the set

$$\left\{ \frac{d}{dt} \int_0^{t+\theta} S_0(t+\theta-s)F(y_\lambda(s))\, ds, \quad \lambda \in \Lambda \right\}.$$

Using the Arzela-Ascoli theorem, we obtain the result of Theorem 29 ■

Consider now the linear autonomous functional differential equation

$$\begin{cases} u'(t) = A_0 u(t) + L(u_t), & t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{15}$$

where $L$ is a continuous linear functional from $\mathcal{C}_X$ into $X$.

Given $\varphi \in \mathcal{C}_X$. It is clear that there exists a unique function $z := z(., \varphi) : [0, +\infty) \to \mathcal{C}_X$ which solves the following abstract integral equation

$$z(t) = S(t)\varphi + \frac{d}{dt} \left( \int_0^t \widetilde{S}(t-s) X_0 L(z(s))\, ds \right), \qquad \text{for } t \geq 0. \tag{16}$$

**Theorem 30** *The family of operators* $(U(t))_{t \geq 0}$ *defined on* $\mathcal{C}_X$ *by*

$$U(t)\varphi = z(t, \varphi),$$

*is a locally Lipschitz continuous integrated semigroup on* $\mathcal{C}_X$ *generated by the operator* $\mathrm{P}$ *defined by*

$$\begin{cases} D(\mathrm{P}) = & \left\{ \varphi \in C^1([-r, 0], X);\ \varphi(0) \in D(A_0), \right. \\ & \quad \left. \varphi'(0) = A_0 \varphi(0) + L(\varphi) \right\}, \\ \mathrm{P}\varphi = & \varphi'. \end{cases}$$

**Proof.** Consider the operator

$$\widetilde{L} : \mathcal{C}_X \to \widetilde{\mathcal{C}}_X$$

defined by

$$\widetilde{L}(\varphi) = X_0 L(\varphi).$$

Using a result of Kellermann [51], one can prove that the operator $\widetilde{G}$ defined in $\widetilde{\mathcal{C}}_X$ by

$$\begin{cases} D(\widetilde{G}) = & D(\widetilde{A}), \\ \widetilde{G} \quad = & \widetilde{A} + \widetilde{L}, \end{cases}$$

where $\widetilde{A}$ is defined in Proposition 15, is the generator of a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathcal{C}}_X$, because $\widetilde{A}$ satisfies (**HY**) and $\widetilde{L} \in \mathcal{L}(\mathcal{C}_X, \widetilde{\mathcal{C}}_X)$.

Let us introduce the part $G$ of $\widetilde{G}$ in $\mathcal{C}_X$, which is defined by :

$$\begin{cases} D(G) = & \left\{ \varphi \in D(\widetilde{G});\ \ \widetilde{G}\varphi \in \mathcal{C}_X \right\}, \\ G(\varphi) = & \widetilde{G}(\varphi). \end{cases}$$

It is easy to see that

$$G = P.$$

Then, $P$ is the generator of a locally Lipschitz continuous integrated semigroup $(V(t))_{t \geq 0}$ on $\mathcal{C}_X$.

On the other hand, if we consider, for each $\varphi \in \mathcal{C}_X$, the nonhomogeneous Cauchy problem

$$\begin{cases} \dfrac{dz}{dt}(t) & = \widetilde{A}z(t) + h(t), \text{ for } t \geq 0, \\ z(0) & = 0, \end{cases} \tag{17}$$

where $h : [0, +\infty[ \to \widetilde{\mathcal{C}}_X$ is given by

$$h(t) = \varphi + \widetilde{L}(V(t)\varphi).$$

By Theorem 13, the nonhomogeneous Cauchy problem (17) has a unique integral solution $z$ given by

$$\begin{aligned} z(t) = & \ \frac{d}{dt}\left( \int_0^t \widetilde{S}(t-s)h(s)\, ds \right), \\ = & \ \frac{d}{dt}\left( \int_0^t \widetilde{S}(t-s)\varphi\, ds \right) + \frac{d}{dt}\left( \int_0^t \widetilde{S}(t-s)X_0 L\left(V(s)\varphi\right)\, ds \right). \end{aligned}$$

Then,

$$z(t) = S(t)\varphi + \frac{d}{dt}\left( \int_0^t \widetilde{S}(t-s)X_0 L\left(V(s)\varphi\right)\, ds \right).$$

On the other hand, Proposition 6 gives

$$V(t)\varphi = P\left(\int_0^t V(s)\varphi\, ds\right) + t\varphi.$$

Moreover, for $\psi \in D(P)$, we have

$$P\psi = \widetilde{A}\psi + X_0 L(\psi).$$

Then, we obtain

$$V(t)\varphi = \widetilde{A}\left(\int_0^t V(s)\varphi\, ds\right) + X_0 L\left(\int_0^t V(s)\varphi\, ds\right) + t\varphi.$$

So,

$$V(t)\varphi = \widetilde{A}\left(\int_0^t V(s)\varphi\, ds\right) + \int_0^t h(s)\, ds.$$

Hence, the function $t \to V(t)\varphi$ is an integral solution of Equation (17). By uniqueness, we conclude that $V(t)\varphi = z(t)$, for all $t \geq 0$. Then we have $U(t) = V(t)$ on $\mathcal{C}_X$. Thus the proof of Theorem 30 ■

Let $\mathcal{B}$ be the part of the operator $P$ on $E$. Then,

$$\begin{cases} D(\mathcal{B}) = & \left\{\varphi \in \mathcal{C}^1([-r,0], X);\ \varphi(0) \in D(A_0),\ \varphi'(0) \in \overline{D(A_0)}\right. \\ & \quad \left.\varphi'(0) = A_0\varphi(0) + L(\varphi)\right\}, \\ \mathcal{B}\varphi = & \varphi'. \end{cases}$$

**Corollary 31** *$\mathcal{B}$ is the infinitesimal generator of the $C_0-$semigroup $(U'(t))_{t\geq 0}$ on $E$.*

**Proof.** See [64] ■

**Corollary 32** *Under the same assumptions as in Theorem 29, the linear $C_0-$semigroup $(U'(t))_{t\geq 0}$ is compact on $E$, for every $t > r$.*

To define a fundamental integral solution $Z(t)$ associated to Equation (15), we consider, for $\widetilde{\varphi} \in \widetilde{\mathcal{C}}_X$, the integral equation

$$z(t) = \widetilde{S}(t)\widetilde{\varphi} + \frac{d}{dt}\left(\int_0^t \widetilde{S}(t-s) X_0 L(z(s))\, ds\right), \quad \text{for } t \geq 0. \tag{18}$$

One can show the following result.

**Proposition 33** *Given $\widetilde{\varphi} \in \widetilde{\mathcal{C}}_X$, the abstract integral equation (18) has a unique solution $z := z(., \widetilde{\varphi})$ which is a continuous mapping from $[0, +\infty) \to \mathcal{C}_X$. Moreover, the family of operators $\left(\widetilde{U}(t)\right)_{t \geq 0}$ defined on $\widetilde{\mathcal{C}}_X$ by*

$$\widetilde{U}(t)\widetilde{\varphi} = z(t, \widetilde{\varphi})$$

*is a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathcal{C}}_X$ generated by the operator $\widetilde{G}$ defined by*

$$\begin{cases} D(\widetilde{G}) = & D(\widetilde{A}), \\ \widetilde{G}\varphi = & \widetilde{A} + X_0 L(\varphi). \end{cases}$$

For each complex number $\lambda$, we define the linear operator

$$\Delta(\lambda) : D(A_0) \to X$$

by

$$\Delta(\lambda)x := \lambda x - A_0 x - L\left(e^{\lambda \cdot} x\right), \qquad x \in D(A_0).$$

where $e^{\lambda \cdot} x : [-r, 0] \to \mathcal{C}_X$, is defined for $x \in X$ by (note that we consider here the comlexification of $\mathcal{C}_X$)

$$\left(e^{\lambda \cdot} x\right)(\theta) = e^{\lambda\theta} x, \qquad\qquad \theta \in [-r, 0].$$

We will call $\lambda$ a characteristic value of Equation (15) if there exists $x \in D(A_0) \backslash \{0\}$ solving the characteristic equation $\Delta(\lambda)x = 0$. The multiplicity of a characteristic value $\lambda$ of Equation (15) is defined as $dim\, Ker \Delta(\lambda)$.

We have the following result.

**Corollary 34** *There exits $\omega \in I\!R$, such that for $\lambda > \omega$ and $c \in X$, one has*

$$(\lambda I - \widetilde{G})^{-1}(X_0 c) = e^{\lambda \cdot} \Delta(\lambda)^{-1} c.$$

**Proof.** We have, for $\lambda > \omega_0$

$$\Delta(\lambda) = (\lambda I - A_0)\left(I - (\lambda I - A_0)^{-1} L(e^{\lambda \cdot} I)\right).$$

Let $\omega > \max(0, \omega_0 + M_0 \|L\|)$ and

$$K_\lambda = (\lambda I - A_0)^{-1} L(e^{\lambda \cdot} I).$$

Then,

$$\|K_\lambda\| \leq \frac{M_0 \|L\|}{\lambda - \omega_0} < 1, \qquad \text{for } \lambda > \omega.$$

Hence the operator $\Delta(\lambda)$ is invertible for $\lambda > \omega$.
Consider the equation

$$(\lambda I - \widetilde{G})(e^{\lambda \cdot} a) = X_0 c,$$

where $c \in X$ is given and we are looking for $a \in D(A_0)$. This yields

$$\lambda e^{\lambda \cdot} c - \lambda e^{\lambda \cdot} c + X_0 \left( \lambda c - A_0 c - L \left( e^{\lambda \cdot} c \right) \right) = X_0 a.$$

Then, for $\lambda > \omega$

$$c = \Delta(\lambda)^{-1} a.$$

Consequently,

$$\left( \lambda I - \widetilde{G} \right)^{-1} (X_0 c) = e^{\lambda \cdot} \Delta(\lambda)^{-1} c \ \blacksquare$$

**Corollary 35** *For each $c \in X$, the function $\widetilde{U}(.)(X_0 c)$ satisfies, for $t \geq 0$ and $\theta \in [-r, 0]$, the following translation property*

$$\left( \widetilde{U}(t)(X_0 c) \right)(\theta) = \begin{cases} \left( \widetilde{U}(t+\theta)(X_0 c) \right)(0), & t + \theta \geq 0, \\ 0, & t + \theta \leq 0. \end{cases}$$

We can consider the following linear operator

$$Z(t) : X \to X$$

defined, for $t \geq 0$ and $c \in X$, by

$$Z(t)c = \widetilde{U}(t)(X_0 c)(0)$$

**Corollary 36** *$Z(t)$ is the fundamental integral solution of Equation (15); that is*

$$\Delta(\lambda)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} Z(t)\, dt, \qquad \text{for } \lambda > \omega.$$

**Proof.** We have, for $c \in X$

$$\left( \lambda I - \widetilde{G} \right)^{-1} (X_0 c) = e^{\lambda \cdot} \Delta(\lambda)^{-1} c.$$

Then

$$e^{\lambda\theta}\Delta(\lambda)^{-1}c = \lambda \int_0^{+\infty} e^{-\lambda s} \left(\widetilde{U}(s)(X_0 c)\right)(\theta)\; ds = \lambda \int_{-\theta}^{+\infty} e^{-\lambda s} Z(s+\theta) c \, ds.$$

So,

$$\Delta(\lambda)^{-1}c = \lambda \int_0^{+\infty} e^{-\lambda t} Z(t) c \, dt \quad \blacksquare\blacksquare$$

It is easy to prove the following result.

**Corollary 37** *If $c \in \overline{D(A_0)}$ then the function*

$$t \to Z(t)c$$

*is differentiable for all $t > 0$ and we have*

$$\Delta(\lambda)^{-1}c = \int_0^{+\infty} e^{-\lambda t} Z'(t) c \, dt.$$

*Hence the name of fundamental integral solution.*

The fundamental solution $Z'(t)$ is defined only for $c \in \overline{D(A_0)}$ and is discontinuous at zero. For these reasons we use the fundamental integral solution $Z(t)$ or equivalently $\widetilde{U}(.)(X_0)$.

We construct now a variation-of-constants formula for the linear nonhomogeneous system

$$u'(t) = A_0 u(t) + L(u_t) + f(t), \quad t \geq 0, \tag{19}$$

or its integrated form

$$y(t) = S'(t)\varphi + \frac{d}{dt}\left(\int_0^t \widetilde{S}(t-s) X_0 L\left[(y(s)) + f(s)\right]\, ds\right), \qquad \text{for } t \geq 0, \tag{20}$$

where $f : [0, +\infty) \to X$ is a continuous function.

**Theorem 38** *For any $\varphi \in \mathcal{C}_X$, such that $\varphi(0) \in \overline{D(A_0)}$, the function $y : [0, +\infty) \to X$ defined by*

$$y(t) = U'(t)\varphi + \frac{d}{dt} \int_0^t \widetilde{U}(t-s) X_0 f(s) \, ds, \tag{21}$$

*satisfies Equation (20).*

**Proof.** It follows immediately from Theorem 18 and Corollary 19 that, for $\varphi \in \mathcal{C}_X$ and $\varphi(0) \in \overline{D(A_0)}$, Equation (20) has a unique solution $y$ which is the integral solution of the equation

$$\begin{cases} y'(t) = \widetilde{A}y(t) + X_0\left[L(y(t)) + f(t)\right], & t \geq 0, \\ y(0) = \varphi. \end{cases}$$

On the other hand, $D(\widetilde{A}) = D(\widetilde{G})$ and $\widetilde{G} = \widetilde{A} + X_0 L$. Then, $y$ is the integral solution of the equation

$$\begin{cases} y'(t) = \widetilde{G}y(t) + X_0 f(t), & t \geq 0, \\ y(0) = \varphi. \end{cases}$$

Moreover, we have

$$\overline{D(\widetilde{G})} = \overline{D(P)} = \left\{\varphi \in \mathcal{C}_X, \quad \varphi(0) \in \overline{D(A_0)}\right\} = E,$$

and the part of $\widetilde{G}$ in $\mathcal{C}_X$ is the operator $P$. Then $t \to U(.)\varphi$ is differentiable in $t \geq 0$ and Theorem 13 implies

$$y(t) = U'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{U}(t-s)X_0 f(s)\, ds \quad \blacksquare\blacksquare$$

# 5 Principle of linearized stability

In this section, we give a result of linearized stability near an equilibrium point of Equation (1) in the autonomous case, that is

$$\begin{cases} \dfrac{du}{dt}(t) = A_0 u(t) + F(u_t), & t \geq 0, \\ u_0 = \varphi. \end{cases} \tag{22}$$

We make the following hypothesis :
$F$ is continuously differentiable, $F(0) = 0$ and $F$ is satisfies the Lipschitz condition

$$|F(\varphi_1) - F(\varphi_2)| \leq L \left\|\varphi_1 - \varphi_2\right\|, \text{ for all } \varphi_1, \varphi_2 \in \mathcal{C}_X,$$

where $L$ is a positive constant.

Let $T(t) : E \to E$, for $t \geq 0$, be defined by

$$T(t)(\varphi) = u_t(., \varphi),$$

where $u(.,\varphi)$ is the unique integral solution of Equation (22) and

$$E = \left\{ \varphi \in \mathcal{C}_X, \quad \varphi(0) \in \overline{D(A_0)} \right\}.$$

We know that the family $(T(t))_{t\geq 0}$ is a nonlinear strongly continuous semigroups of continuous operators on $E$.

Consider the linearized equation of (22) corresponding to the Fréchet-derivative $D_\varphi F(0) = F'(0)$ at 0 :

$$\begin{cases} \dfrac{du}{dt}(t) = A_0 u(t) + F'(0)u_t, & t \geq 0, \\ u_0 = \varphi \in \mathcal{C}_X, \end{cases} \tag{23}$$

and let $(U'(t))_{t\geq 0}$ be the corresponding linear $C_0-$semigroup on $E$.

**Proposition 39** *The Fréchet-derivative at zero of the nonlinear semigroup $T(t)$, $t \geq 0$, associated to Equation (22), is the linear semigroup $U'(t)$, $t \geq 0$, associated to Equation (23).*

**Proof.** It suffices to show that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| T(t)(\varphi) - U'(t)\varphi \right\| \leq \varepsilon \left\| \varphi \right\|, \text{ for } \left\| \varphi \right\| \leq \delta.$$

We have

$$\begin{aligned} \| T(t)(\varphi) - U'(t)\varphi \| &= \theta \in [-r,0] \sup \left| (T(t)(\varphi))(\theta) - (U'(t)\varphi)(\theta) \right|, \\ &= t+\theta \geq 0 \theta \in [-r,0] \sup \left| (T(t)(\varphi))(\theta) - (U'(t)\varphi)(\theta) \right|, \end{aligned}$$

and, for $t + \theta \geq 0$

$$(T(t)(\varphi))(\theta) - \left(U'(t)\varphi\right)(\theta) = \frac{d}{dt} \int_0^{t+\theta} S_0(t+\theta-s) \left( F(T(s)(\varphi)) - F'(0)(U'(s)\varphi) \right) ds.$$

It follows that

$$\left\| T(t)(\varphi) - U'(t)\varphi \right\| \leq M_0 e^{\omega_0 t} \int_0^t e^{-\omega_0 s} \left| F(T(s)(\varphi)) - F'(0)(U'(s)\varphi) \right| ds$$

and

$$\begin{aligned} \| T(t)(\varphi) - U'(t)\varphi \| \leq \quad & M_0 e^{\omega_0 t} \left( {}_0^t e^{-\omega_0 s} \left| F(T(s)(\varphi)) - F(U'(s)\varphi) \right| ds \right. \\ & \left. + {}_0^t e^{-\omega_0 s} \left| F(U'(s)\varphi) - F'(0)(U'(s)\varphi) \right| ds \right). \end{aligned}$$

By virtue of the continuous differentiability of $F$ at 0, we deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^t e^{-\omega_0 s} \left| F(U'(s)\varphi) - F'(0)(U'(s)\varphi) \right| ds \leq \varepsilon \left\| \varphi \right\|, \quad \text{for } \left\| \varphi \right\| \leq \delta.$$

On the other hand, we obtain

$$\int_0^t e^{-\omega_0 s} \left| F(T(s)(\varphi)) - F(U'(s)\varphi) \right| ds \leq L \int_0^t e^{-\omega_0 s} \left\| T(s)(\varphi) - U'(s)\varphi \right\| ds.$$

Consequently,

$$\left\| T(t)(\varphi) - U'(t)\varphi \right\| \leq M_0 e^{\omega_0 t} \left( \varepsilon \left\| \varphi \right\| + L \int_0^t e^{-\omega_0 s} \left\| T(s)(\varphi) - U'(s)\varphi \right\| ds \right).$$

By Gronwall's Lemma, we obtain

$$\left\| T(t)(\varphi) - U'(t)\varphi \right\| \leq M_0 \varepsilon \left\| \varphi \right\| e^{(LM_0 + \omega_0)t}.$$

We conclude that $T(t)$ is differentiable at 0 and $D_\varphi T(t)(0) = U'(t)$, for each $t \geq 0$ ■

**Definition 40** *Let $Y$ be a Banach space, $(V(t))_{t\geq 0}$ a strongly continuous semigroup of operators $V(t) : W \subseteq Y \to W$, $t \geq 0$, and $x_0 \in W$ an equilibrium of $(V(t))_{t\geq 0}$ (i.e., $V(t)x_0 = x_0$, for all $t \geq 0$). The equilibrium $x_0$ is called exponentially asymptotically stable if there exist $\delta > 0, \mu > 0, k \geq 1$ such that*

$$\left\| V(t)x - x_0 \right\| \leq k e^{-\mu t} \left\| x - x_0 \right\| \text{ for all } x \in W \text{ with } \left\| x - x_0 \right\| \leq \delta \text{ and all } t \geq 0.$$

We have the following result.

**Theorem 41** *Suppose that the zero equilibrium of $(U'(t))_{t\geq 0}$ is exponentially asymptotically stable, then zero is exponentially asymptotically stable equilibrium of $(T(t))_{t\geq 0}$.*

The proof of this theorem is based on the following result.

**Theorem 42** *(Desh and Schappacher [41]) Let $(V(t))_{t\geq 0}$ be a nonlinear strongly continuous semigroup of type $\gamma$ on a subset $W$ of a Banach space $Y$, i.e.*

$$\left\| V(t)x_1 - V(t)x_2 \right\| \leq M' e^{\gamma t} \left\| x_1 - x_2 \right\|, \text{ for all } x_1, x_2 \in W$$

*and assume that $x_0 \in W$ is an equilibrium of $(V(t))_{t\geq 0}$ such that $V(t)$ is Fréchet-differentiable at $x_0$ for each $t \geq 0$, with $Y(t)$ the Fréchet-derivative at $x_0$ of $V(t)$, $t \geq 0$. Then, $(Y(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $Y$. If the zero equilibrium of $(Y(t))_{t\geq 0}$ is exponentially asymptotically stable, then $x_0$ is an exponentially asymptotically stable equilibrium of $(V(t))_{t\geq 0}$.*

# 6 Spectral Decomposition

In this part, we show in the linear autonomous case (Equation (15)), the existence of a direct sum decomposition of the state space

$$E = \left\{ \varphi \in \mathcal{C}_X, \quad \varphi(0) \in \overline{D(A_0)} \right\}$$

into three subspaces : stable, unstable and center, which are semigroup invariants. We assume that the semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A_0)}$ is compact. It follows from the compactness property of the semigroup $U'(t)$, for $t > r$, on $E$, the following results.

**Corollary 43** *[60] For each $t > r$, the spectrum $\sigma(U'(t))$ is a countable set and is compact with the only possible accumulation point $0$ and if $\mu \neq 0 \in \sigma(U'(t))$ then $\mu \in P\sigma(U'(t))$, (where $P\sigma(U'(t))$ denotes the point spectrum).*

**Corollary 44** *[60] There exists a real number $\delta$ such that $\operatorname{Re}\lambda \leq \delta$ for all $\lambda \in \sigma(\mathcal{B})$. Moreover, if $\beta$ is a given real number then there exists only a finite number of $\lambda \in P\sigma(\mathcal{B})$ such that $\operatorname{Re}\lambda > \beta$.*

We can give now an exponential estimate for the semigroup solution.

**Proposition 45** *Assume that $\delta$ is a real number such that $\operatorname{Re}\lambda \leq \delta$ for all characteristic values of Equation (15). Then, for $\gamma > 0$ there exists a constant $k(\gamma) \geq 1$ such that*

$$\left\| U'(t)\varphi \right\| \leq k(\gamma) e^{(\delta+\gamma)t} \left\| \varphi \right\|, \qquad \textit{for all } t \geq 0,\ \varphi \in E.$$

**Proof.** Let $\omega_1$ be defined by

$$\omega_1 := \inf\left\{ \omega \in \mathbb{R} : \qquad t \geq 0 \sup\left( e^{-\omega t} \left\| U'(t) \right\| \right) < +\infty \right\}.$$

The compactness property of the semigroup (see [55]) implies that

$$\omega_1 = s_1(\mathcal{B}) := \sup\left\{ \operatorname{Re}\lambda : \quad \lambda \in P\sigma(\mathcal{B}) \right\}.$$

On the other hand, if $\lambda \in P\sigma(\mathcal{B})$ then there exists $\varphi \neq 0 \in D(\mathcal{B})$ such that $\mathcal{B}\varphi = \lambda\varphi$. This implies that

$$\varphi(\theta) = e^{\lambda t}\varphi(0) \quad \text{and} \quad \varphi'(0) = A_0\varphi(0) - L(\varphi) \text{ with } \varphi(0) \neq 0.$$

Then, $\Delta(\lambda)\varphi(0) = 0$. We deduce that $\lambda$ is a characteristic value of Equation (15).

We will prove now the existence of $\lambda \in P\sigma(\mathcal{B})$ such that $\operatorname{Re}\lambda = s_1(\mathcal{B})$. Let $(\lambda_n)_n$ be a sequence in $P\sigma(\mathcal{B})$ such that $\operatorname{Re}\lambda_n \to s_1(\mathcal{B})$ as $n \to +\infty$. Then, there exists $\beta$ such that $\operatorname{Re}\lambda_n > \beta$ for $n \geq n_0$ with $n_0$ large enough. From Corollary 44, we deduce that $\{\lambda_n : \operatorname{Re}\lambda_n > \beta\}$ is finite. So, the sequence $(\operatorname{Re}\lambda_n)_n$ is stationary. Consequently, there exists $n$ such that $\operatorname{Re}\lambda_n = s_1(\mathcal{B})$. This completes the proof of Proposition 45 ■

The asymptotic behavior of solutions can be now completely obtained by the characteristic equation.

**Theorem 46** *Let $\delta$ be the smallest real number such that if $\lambda$ is a characteristic value of Equation (15), then $\operatorname{Re}\lambda \leq \delta$. If $\delta < 0$, then for all $\varphi \in E$, $\|U'(t)\varphi\| \to 0$ as $t \to +\infty$. If $\delta = 0$ then there exists $\varphi \in E\backslash\{0\}$ such that $\|U'(t)\varphi\| = \|\varphi\|$ for all $t \geq 0$. If $\delta > 0$, then there exists $\varphi \in E$ such that $\|U'(t)\varphi\| \to +\infty$ as $t \to +\infty$.*

**Proof.** Assume that $\delta < 0$, then we have $\omega_1 = s_1(\mathcal{B}) < 0$ and the stability holds. If $\delta = 0$, then there exists $x \neq 0$ and a complex $\lambda$ such that $\operatorname{Re}\lambda = 0$ and $\Delta(\lambda)x = 0$. Then, $\lambda \in P\sigma(\mathcal{B})$ and $e^{\lambda t} \in P\sigma(U'(t))$. Consequently, there exists $\varphi \neq 0$ such that

$$U'(t)\varphi = e^{\lambda t}\varphi.$$

This implies that $\|U'(t)\varphi\| = \left\|e^{\lambda t}\varphi\right\| = \|\varphi\|$. Assume now that $\delta > 0$. Then, there exists $x \neq 0$ and a complex $\lambda$ such that $\operatorname{Re}\lambda = \delta$ and $\Delta(\lambda)x = 0$. Then, there exists $\varphi \neq 0$ such that $\|U'(t)\varphi\| = e^{\delta t}\|\varphi\| \to +\infty$, as $t \to +\infty$. This completes the proof of Theorem 46 ■

Using the same argument as in [67], [section 3.3, Theorem 3.1], we obtain the following result.

**Theorem 47** *Suppose that $X$ is complex. Then, there exist three linear subspaces of $E$ denoted by $S$, $US$ and $CN$, respectively, such that*

$$E = S \oplus US \oplus CN$$

*and*
**(i)** $\mathcal{B}(S) \subset S$, $\mathcal{B}(US) \subset US$ *and* $\mathcal{B}(CN) \subset CN$;

(**ii**) *$US$ and $CN$ are finite dimensional;*
(**iii**) $P\sigma(\mathcal{B}\mid_{US}) = \{\lambda \in P\sigma(\mathcal{B}) : \operatorname{Re}\lambda > 0\}$, $P\sigma(\mathcal{B}\mid_{CN}) = \{\lambda \in P\sigma(\mathcal{B}) : \operatorname{Re}\lambda = 0\}$;
(**iv**) $U'(t)(US) \subset US$, $U'(t)(CN) \subset CN$, *for* $t \in \mathbb{R}$, $U'(t)(S) \subset S$, *for* $t \geq 0$;
(**v**) *for any* $0 < \gamma < \alpha := \inf\{|\operatorname{Re}\lambda| : \lambda \in P\sigma(\mathcal{B}) \text{ and } \operatorname{Re}\lambda \neq 0\}$, *there exists* $M = M(\gamma) > 0$ *such that*

$$\begin{aligned}
&\|U'(t)P_{US}\varphi\| \leq Me^{\gamma t}\|P_{US}\varphi\|, \qquad t \leq 0,\\
&\|U'(t)P_{CN}\varphi\| \leq Me^{\frac{\gamma}{3}t}\|P_{CN}\varphi\|, \qquad t \in \mathbb{R},\\
&\|U'(t)P_{S}\varphi\| \leq Me^{-\gamma t}\|P_{S}\varphi\|, \qquad t \geq 0,
\end{aligned}$$

*where $P_S, P_{US}$ and $P_{CN}$ are projections of $E$ into $S$, $US$ and $CN$ respectively. $S$, $US$ and $CN$ are called stale, unstable and center subspaces of the semigroup $(U'(t))_{t\geq 0}$.*

## 7 Existence of bounded solutions

We reconsider now the equation (19) mentioned in the introduction. For the convenience of the reader, we restate this equation

$$\begin{cases} \dfrac{du}{dt}(t) = A_0u(t) + L(u_t) + f(t), & t \geq 0,\\ u_0 = \varphi \in \mathcal{C}_X, \end{cases}$$

and its integrated form

$$u_t = U'(t)\varphi + \frac{d}{dt}\int_0^t \widetilde{U}(t-s)X_0f(s)ds, \quad \text{for } t \geq 0. \tag{24}$$

where $f$ is a continuous function form $\mathbb{R}$ into $X$.

Thanks to Lemma 17, we will use the following integrated form of Equation (19), which is equivalent to (24) :

$$u_t = U'(t)\varphi + \lambda \to +\infty\lim \int_0^t U'(t-s)\widetilde{B_\lambda}X_0f(s)ds, \quad \text{for } t \geq 0, \tag{25}$$

where the operator $\widetilde{B_\lambda} : \langle X_0 \rangle \to \mathcal{C}_X$ is defined by

$$\widetilde{B_\lambda}X_0c = \lambda\left(\lambda I - \widetilde{G}\right)^{-1}(X_0c) = \lambda e^{\lambda\cdot}\Delta(\lambda)^{-1}c, \quad c \in X.$$

We need the following definition.

**Definition 48** *We say that the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{B}) \cap i\mathbb{R} = \emptyset.$$

Theorem 47 implies that, in the hyperbolic case the center subspace $CN$ is reduced to zero. Thus, we have the following result.

**Corollary 49** *If the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic, then the space $E$ is decomposed as*

$$E = S \oplus US$$

*and there exist positive constants $\overline{M}$ and $\gamma$ such that*

$$\begin{array}{ll} \|U'(t)\varphi\| \leq \overline{M}e^{-\gamma t}\|\varphi\|, & t \geq 0,\ \varphi \in S, \\ \|U'(t)\varphi\| \leq \overline{M}e^{\gamma t}\|\varphi\|, & t \leq 0,\ \varphi \in US. \end{array}$$

We give now the first main result of this section.

**Theorem 50** *Assume that the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic. Let $B$ represent $B(\mathbb{R}^-)$, $B(\mathbb{R}^+)$ or $B(\mathbb{R})$, the set of bounded continuous functions from $\mathbb{R}^-$, $\mathbb{R}^+$ or $\mathbb{R}$ respectively to $X$. Let $\pi : B \to B$ be a projection onto the integral solutions of Equation (15) (for any $\varphi \in E$) which are in $B$. Then, for any $f \in B$, there is a unique solution $\mathcal{K}f \in B$ of Equation (25) (for some $\varphi \in E$) such that $\pi\mathcal{K}f = 0$ and $\mathcal{K} : B \to B$ is a continuous linear operator. Moreover,*
**(i)** *for $B = B(\mathbb{R}^-)$, we have*

$$\pi(B) = \begin{array}{l} \{x : \mathbb{R}^- \to X, \text{ there exists } \varphi \in US \\ \qquad \text{such that } x(t) = (U'(t)\varphi)(0), \quad t \leq 0\} \end{array}$$

*and*

$$(\mathcal{K}f)_t = \begin{array}{l} s \to -\infty\lim\lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^S d\tau + \\ \lambda \to +\infty\lim \int_0^t U'(t-\tau)\left(\widetilde{B_\lambda}X_0 f(\tau)\right)^{US} d\tau. \end{array}$$

**(ii)** *For $B = B(\mathbb{R}^+)$, we have*

$$\pi(B) = \begin{array}{l} \{x : \mathbb{R}^+ \to X, \text{ there exists } \varphi \in S \\ \qquad \text{such that } x(t) = (U'(t)\varphi)(0), \quad t \geq 0\} \end{array}$$

*and*

$$
\begin{aligned}
(\mathcal{K}f)_t = \quad & \lambda \to +\infty\lim \int_0^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S d\tau + \\
& s \to +\infty\lim \lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US} d\tau.
\end{aligned}
$$

**(iii)** *For $B = B(\mathbb{R})$, we have*

$$\pi(B) = \{0\}$$

*and*

$$
\begin{aligned}
(\mathcal{K}f)_t = \quad & s \to -\infty\lim \lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S d\tau + \\
& s \to +\infty\lim \lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US} d\tau.
\end{aligned}
$$

**Proof.** Theorem 47 implies that the space $US$ is finite dimensional and $U'(t)\,(US) \subseteq US$. Then,
$\left\{x : \mathbb{R}^- \to X, \text{ there exists } \varphi \in US \text{ such that } x(t) = (U'(t)\varphi)\,(0),\ \ t \le 0\right\}$
$\subseteq \pi(B(\mathbb{R}^-))$. Conversely, let $\varphi \in S$ and $u(.,\varphi)$ be the integral solution of Equation (15) in $S$, which is bounded on $\mathbb{R}^-$. Assume that there is a $t \in (-\infty, 0]$ such that $u_t(.,\varphi) \neq 0$. Then, for any $s \in (-\infty, t)$, we have

$$u_t(.,\varphi) = U'(t-s)u_s(.,\varphi).$$

Thanks to Corollary 49 we have

$$\|u_t(.,\varphi)\| \le \overline{M} e^{-\gamma(t-s)} \|u_s(.,\varphi)\|, \quad s \le t.$$

Since $u_s(.,\varphi)$ is bounded, we deduce that $u_s(.,\varphi) = 0$. Therefore,

$$
\begin{aligned}
\pi(B(\mathbb{R}^-)) \subseteq \quad & \{x : \mathbb{R}^- \to X, \text{ there exists } \varphi \in US \\
& \qquad \text{such that } x(t) = (U'(t)\varphi)\,(0), \quad t \le 0\}.
\end{aligned}
$$

In the same manner, one can prove the same relations for $B(\mathbb{R}^+)$ and $B(\mathbb{R})$.

Let $f \in B(\mathbb{R}^-)$ and $u = u(.,\varphi, f)$ be a solution of Equation (25) in $B(\mathbb{R}^-)$, with initial value $\varphi \in E$. Then, the function $u$ can be decomposed as

$$u_t = u_t^{US} + u_t^S,$$

where $u_t^{US} \in US$ and $u_t^S \in S$ are given by

$$u_t^{US} = U'(t-s)u_s^{US} + \lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US} d\tau, \quad \text{for } t, s \in \mathbb{R}, \tag{26}$$

$$u_t^S = U'(t-s)u_s^S + \lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S d\tau, \quad \text{for } s \le t \le 0, \tag{27}$$

since $U'(t)$ is defined on $US$ for all $t \in \mathbb{R}$. By Corollary 49, we deduce that

$$u_t^S = s \to -\infty\lim\lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S d\tau, \quad \text{for } t \le 0. \tag{28}$$

By Lemma 16, we get $\widetilde{B_\lambda} X_0 = \lambda e^{\lambda \cdot}\Delta^{-1}(\lambda)$. Thus,

$$\left\| U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S \right\| \le \overline{M} e^{-\gamma(t-\tau)} \frac{M\lambda}{\lambda - \omega_1} \tau \in (-\infty, 0]\sup \|f(\tau)\|.$$

Consequently, we get

$$\left\|u_t^S\right\| \le \frac{M\overline{M}}{\gamma} \tau \in (-\infty, 0]\sup \|f(\tau)\|, \qquad t \le 0. \tag{29}$$

We have proved that

$$\begin{aligned} u_t = \quad & U'(t)\varphi^{US} + \lambda \to +\infty\lim \textstyle\int_0^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US} d\tau + \\ & s \to -\infty\lim\lambda \to +\infty\lim \textstyle\int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S d\tau, \qquad \text{for } t \le 0. \end{aligned} \tag{30}$$

We obtain also, for $t \le 0$, the following estimate

$$\begin{aligned} & \left\| U'(t)\varphi^{US} + \lambda \to +\infty\lim \textstyle\int_0^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US} d\tau \right\| \le \\ & \qquad \overline{M} e^{\gamma t} \left\|\varphi^{US}\right\| + \tfrac{M\overline{M}}{\gamma} \tau \in (-\infty, 0]\sup \|f(\tau)\|. \end{aligned} \tag{31}$$

Conversely, we can verify that the expression (30) is a solution of Equation (25) in $B(\mathbb{R}^-)$ satisfying the estimates (29) and (31) for every $\varphi \in E$.

Let $u = u(., \varphi^{US}, f)$ be defined by (30) and let $\mathcal{K} : B(\mathbb{R}^-) \to B(\mathbb{R}^-)$ be defined by $\mathcal{K}f = (I - \pi)\, u(., 0, f)$. We can easily verify that

$$\begin{cases} u_t(., \varphi^{US}, 0) = U'(t)\varphi^{US}, \\ (I-\pi)\, u(., \varphi^{US}, 0) = 0, \\ u(., \varphi^{US}, f) = u(., \varphi^{US}, 0) + u(., 0, f). \end{cases}$$

Therefore, $\mathcal{K}$ is a continuous linear operator on $B(\mathbb{R}^-)$, $\mathcal{K}f$ satisfies Equation (25) for every $f \in B(\mathbb{R}^-)$, $\pi\mathcal{K} = 0$ and $\mathcal{K}$ is explicitly given by Theorem 50.

For $B(\mathbb{R}^+)$ and $B(\mathbb{R})$ the proofs are similar. This completes the proof of Theorem 50 ■

We consider now the nonlinear equation

$$\frac{du}{dt}(t) = A_0 u(t) + L(u_t) + g(t, u_t), \tag{32}$$

and its integrated form

$$u_t = U'(t)\varphi + \lambda \to +\infty \lim \int_0^t U'(t-s)\widetilde{B_\lambda} X_0 g(s, u_s) ds, \tag{33}$$

where $g$ is continuous from $\mathbb{R} \times \mathcal{C}_X$ into $X$. We will assume that
(**H1**) $g(t, 0) = 0$ for $t \in \mathbb{R}$ and there exists a nondecreasing function $\alpha$ : $[0, +\infty) \to [0, +\infty)$ with $h \to 0 \lim \alpha(h) = 0$ and
$\|g(t, \varphi_1) - g(t, \varphi_2)\| \leq \alpha(h) \|\varphi_1 - \varphi_2\|$ for $\varphi_1, \varphi_2 \in E$, $\|\varphi_1\|, \|\varphi_2\| \leq h$ and $t \in \mathbb{R}$.

**Proposition 51** *Assume that the semigroup $(U'(t))_{t \geq 0}$ is hyperbolic and the assumption (**H1**) holds. Then, there exists $h > 0$ and $\varepsilon \in ]0, h[$ such that for any $\varphi \in S$ with $\|\varphi\| \leq \varepsilon$, Equation (33) has a unique bounded solution $u : [-r, +\infty) \to X$ with $\|u_t\| \leq h$ for $t \geq 0$ and $u_0^S = \varphi$.*

**Proof.** Let $\varphi \in S$. By Theorem 50, it suffices to establish the existence of a bounded solution $u : [-r, +\infty) \to X$ of the following equation

$$\begin{aligned} u_t = \quad & U'(t)\varphi + \lambda \to +\infty \lim \textstyle\int_0^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^S d\tau + \\ & s \to +\infty \lim \lambda \to +\infty \lim \textstyle\int_s^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^{US} d\tau. \end{aligned}$$

Let $(u^{(n)})_{n \in \mathbb{N}}$ be a sequence of continuous functions from $[-r, +\infty)$ to $X$, defined by

$$\begin{aligned} u_t^{(0)} = \quad & U'(t)\varphi \\ u_t^{(n+1)} = \quad & U'(t)\varphi + \lambda \to +\infty \lim \textstyle\int_0^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^S d\tau + \\ & s \to +\infty \lim \lambda \to +\infty \lim \textstyle\int_s^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^{US} d\tau. \end{aligned}$$

It is clear that $\left(u_0^{(n)}\right)^S = \varphi$. Moreover, we can choose $h > 0$ and $\varepsilon \in ]0, h[$ small enough such that, if $\|\varphi\| \leq \varepsilon$ then $\left\|u_t^{(n)}\right\| < h$ for $t \geq 0$.

On the other hand, we have

$$\begin{aligned}\left\|u_t^{(n+1)} - u_t^{(n)}\right\| \quad & \leq \int_0^t \overline{M} M e^{-\gamma(t-\tau)} \alpha(h) \left\|u_\tau^{(n)} - u_\tau^{(n-1)}\right\| d\tau + \\ & \int_t^{+\infty} \overline{M} M e^{\gamma(t-\tau)} \alpha(h) \left\|u_\tau^{(n)} - u_\tau^{(n-1)}\right\| d\tau.\end{aligned}$$

So, by induction we get

$$\left\|u_t^{(n+1)} - u_t^{(n)}\right\| \leq 2h \left(\frac{2\overline{M} M \alpha(h)}{\gamma}\right)^n, \quad t \geq 0.$$

We choose $h > 0$ such that

$$\frac{2\overline{M} M \alpha(h)}{\gamma} < \frac{1}{2}.$$

Consequently, the limit $u := n \to +\infty \lim u^{(n)}$ exists uniformly on $[-r, +\infty)$ and $u$ is continuous on $[-r, +\infty)$. Moreover, $\|u_t\| < h$ for $t \geq 0$ and $u_0^S = \varphi$.

In order to prove that $u$ is a solution of Equation (33), we introduce the following notation

$$\begin{aligned} v(t) = \quad & \Bigg\| u_t - U'(t)\varphi - \lambda \to +\infty \lim \int_0^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^S d\tau \\ & -s \to +\infty \lim \lambda \to +\infty \lim \int_s^t U'(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^{US} d\tau \Bigg\|. \end{aligned}$$

We obtain

$$\begin{aligned} v(t) \leq \quad & \left\|u_t - u_t^{(n+1)}\right\| \\ & + \left\| \lambda \to +\infty \lim \int_0^t U'(t-\tau) \left( \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^S - \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^S \right) d\tau \right\| \\ & + \left\| s \to +\infty \lim \lambda \to +\infty \lim \int_s^t U'(t-\tau) \left( \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau)\right)^{US} - \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^{US} \right) d\tau \right\|. \end{aligned}$$

Moreover, we have

$$u_t - u_t^{(n+1)} = k = n + 1 \sum^{+\infty} \left(u_t^{(k+1)} - u_t^{(k)}\right).$$

It follows that

$$v(t) \leq 2h \left[1 + \frac{2\overline{M} M \alpha(h)}{\gamma}\right] k = n + 1 \sum^{+\infty} \left(\frac{2\overline{M} M \alpha(h)}{\gamma}\right)^k.$$

Consequently, $v = 0$ on $[0, +\infty)$.

To show the uniqueness suppose that $w$ is also a solution of Equation (33) with $\|w_t\| < h$ for $t \geq 0$. Then,

$$\begin{aligned}
&\left\|w_t - u_t^{(n+1)}\right\| \leq \\
&\quad \left\|\lambda \to +\infty\lim \int_0^t U'(t-\tau)\left(\left(\widetilde{B_\lambda} X_0 g(\tau, w_\tau)\right)^S - \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^S\right) d\tau\right\| \\
&+ \left\|s \to +\infty\lim\lambda \to +\infty\lim \int_s^t U'(t-\tau)\left(\left(\widetilde{B_\lambda} X_0 g(\tau, w_\tau)\right)^{US} - \left(\widetilde{B_\lambda} X_0 g(\tau, u_\tau^{(n)})\right)^{US}\right) d\tau\right\|.
\end{aligned}$$

This implies

$$\left\|w_t - u_t^{(n+1)}\right\| \leq 2h\left(\frac{2\overline{M}M\alpha(h)}{\gamma}\right)^n \to 0, \quad \text{as } n \to +\infty.$$

This proves the uniqueness and completes the proof ■

# 8 Existence of periodic or almost periodic solutions

In this section, we are concerned with the existence of periodic (or almost periodic) solutions of Equations (25) and (33).

As a consequence of the existence of a bounded solution we obtain the following result.

**Corollary 52** *Assume that the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic. If the function $f$ is $\omega-$periodic, then the bounded solution of Equation (25) given by Theorem 50 is also $\omega-$periodic.*

**Proof.** Let $u$ be the unique bounded solution of Equation (25). The function $u(. + \omega)$ is also a bounded solution of Equation (25). The uniqueness property implies that $u = u(.+\omega)$. This completes the proof of the corollary ■

We are concerned now with the existence of almost periodic solution of Equation (25). We first recall a definition.

**Definition 53** *A function $g \in B(\mathbb{R})$ is said to be almost periodic if and only if the set*

$$\{g_\sigma : \sigma \in \mathbb{R}\},$$

*where $g_\sigma$ is defined by $g_\sigma(t) = g(t+\sigma)$, for $t \in \mathbb{R}$, is relatively compact in $B(\mathbb{R})$.*

**Theorem 54** *Assume that the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic. If the function $f$ is almost periodic, then the bounded solution of Equation (25) is also almost periodic.*

**Proof.** Let $AP(\mathbb{R})$ be the Banach space of almost periodic functions from $\mathbb{R}$ to $X$ endowed with the uniform norm topology. Define the operator $Q$ by

$$\begin{aligned}(Qf)(t) &= s \to -\infty \lim \lambda \to +\infty \lim \textstyle\int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^S(0)\, d\tau \\ &+ s \to +\infty \lim \lambda \to +\infty \lim \textstyle\int_s^t U'(t-\tau)\left(\widetilde{B_\lambda} X_0 f(\tau)\right)^{US}(0)\, d\tau, \quad \text{for } t \in \mathbb{R}.\end{aligned}$$

Then, $Q$ is a bounded linear operator from $AP(\mathbb{R})$ into $B(\mathbb{R})$. By a sample computation, we obtain

$$(Qf)_\sigma = Q(f_\sigma), \quad \text{for } \sigma \in \mathbb{R}.$$

By the continuity of the operator $Q$, we deduce that $Q(\{f_\sigma : \sigma \in \mathbb{R}\})$ is relatively compact in $B(\mathbb{R})$. This implies that if the function $f$ is almost periodic, then the bounded solution of Equation (25) is also almost periodic. This completes the proof of the theorem.

We are concerned now with the existence of almost periodic solutions of Equation (33). We will assume the followings.
(**H2**) $g$ is almost periodic in $t$ uniformly in any compact set of $\mathcal{C}_X$. This means that for each $\varepsilon > 0$ and any compact set $K$ of $\mathcal{C}_X$ there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon$ contains a number $\tau$ with the property that

$$t \in \mathbb{R}\mathbb{R}, \varphi \in K \sup \|g(t+\tau,\varphi) - g(t,\varphi)\| < \varepsilon.$$

We know from [39] that if the function $g$ is almost periodic in $t$ uniformly in any compact set of $\mathcal{C}_X$ and if $v$ is an almost periodic function, then the function $t \mapsto g(t, v_t)$ is also almost periodic.
(**H3**) $\|g(t,\varphi_1) - g(t,\varphi_2)\| \leq K_1 \|\varphi_1 - \varphi_2\|, \quad t \in \mathbb{R},\ \varphi_1, \varphi_2 \in \mathcal{C}_X.$

We have the following result.

**Proposition 55** *Assume that the semigroup $(U'(t))_{t\geq 0}$ is hyperbolic and* (**H2**), (**H3**) *hold. Then, if in the assumption* (**H3**) *$K_1$ is chosen small enough, Equation (33) has a unique almost periodic solution.*

**Proof.** Consider the operator $H$ defined on $AP(\mathbb{R})$ by

$$Hv = u,$$

where $u$ is the unique almost periodic solution of Equation (25) with $f = g(.,v_.)$. We can see that there exists a positive constant $K_2$ such that

$$\|Hv_1 - Hv_2\| \leq K_1K_2 \|v_1 - v_2\|, \qquad v_1, v_2 \in AP(\mathbb{R}).$$

If $K_1$ is chosen such that $K_1K_2 < 1$, then the map $H$ is a strict contraction in $AP(\mathbb{R})$. So, $H$ has a unique fixed point in $AP(\mathbb{R})$. This gives an almost periodic solution of Equation (33). This completes the proof of the proposition ■

# 9 Applications

We now consider the two examples mentioned in the introduction. For the convenience of the reader, we restate the equations.

**Example 1**

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,a) + \dfrac{\partial u}{\partial a}(t,a) = f(t,a,u_t(\cdot,a)), & t \in [0,T], \ a \in [0,l], \\ u(t,0) = 0, & t \in [0,T], \\ u(\theta,a) = \varphi(\theta,a), & \theta \in [-r,0], \ a \in [0,l]. \end{cases} \tag{34}$$

where $T, l > 0$, $\varphi \in \mathcal{C}_X := \mathcal{C}([-r,0], X)$ and $X = \mathcal{C}([0,l], I\!R)$.

By setting $U(t) = u(t,\cdot)$, Equation (34) reads

$$\begin{cases} V'(t) = A_0V(t) + F(t,V_t), & t \in [0,T], \\ V(0) = \varphi, \end{cases}$$

where $A_0 : D(A_0) \subseteq X \to X$ is the linear operator defined by

$$\begin{cases} D(A_0) = \left\{u \in \mathcal{C}^1([0,l], I\!R); \quad u(0) = 0\right\}, \\ A_0u = -u', \end{cases}$$

and $F : [0,T] \times \mathcal{C}_X \to X$ is the function defined by

$$F(t,\varphi)(a) = f(t,a,\varphi(\cdot,a)), \quad \text{for } t \in [0,T], \ \varphi \in \mathcal{C}_X \text{ and } a \in [0,l].$$

We have $\overline{D(A_0)} = \{u \in \mathcal{C}([0,l], I\!R); \quad u(0) = 0\} \neq X$. Moreover,

$$\begin{cases} \rho(A_0) = \mathbf{C}, \\ \left\|(\lambda I - A_0)^{-1}\right\| \leq \frac{1}{\lambda}, & \text{for } \lambda > 0, \end{cases}$$

this implies that $A_0$ satisfies (**HY**) on $X$ (with $M_0 = 1$ and $\omega_0 = 0$).

We have the following result.

**Theorem 56** *Assume that $F$ is continuous on $[0,T] \times \mathcal{C}_X$ and satisfies a Lipschitz condition*

$$|F(t,\varphi) - F(t,\psi)| \leq L \|\varphi - \psi\|, \quad t \in [0,T] \text{ and } \varphi, \psi \in \mathcal{C}_X,$$

*with $L \geq 0$ constant. Then, for a given $\varphi \in \mathcal{C}_X$, such that*

$$\varphi(0,0) = 0,$$

*there exists a unique function $u : [0,T] \to X$ solution of the following initial value problem*

$$u(t,a) = \begin{cases} \int_0^a f(\tau - a + t\,,\, \tau\,,\, u_{\tau-a+t}(\cdot,\tau))\, d\tau, & \text{if } a \leq t, \\ \varphi(0, a-t) + \int_{a-t}^a f(\tau - a + t\,,\, \tau\,,\, u_{\tau-a+t}(\cdot,\tau))\, d\tau, & \text{if } a \geq t \geq 0, \\ \varphi(t,a), & \text{if } t \leq 0. \end{cases} \quad (35)$$

*Furthermore, $u$ is the unique integral solution of the partial differential equation (34), i.e.*
*(i) $u \in \mathcal{C}_X$, $\int_0^t u(s,\cdot)ds \in \mathcal{C}^1([0,l], I\!R)$ and $\int_0^t u(s,0)ds = 0$, for $t \in [0,T]$,*
*(ii)*

$$u(t,a) = \begin{cases} \varphi(0,a) + \dfrac{\partial}{\partial a} \displaystyle\int_0^t u(s,a)\, ds + \int_0^t f(s,a,u_t(\cdot,a)\, ds, & \text{if } t \in [0,T], \\ \varphi(t,a), & \text{if } t \in [-r,0]. \end{cases}$$

**Proof.** The assumptions of Theorem 56 imply that $\varphi \in \mathcal{C}_X$ and $\varphi(0,\cdot) \in \overline{D(A_0)}$. Consequently, from Theorem 18, we deduce that there exists a unique function $v : [0,T] \to \mathcal{C}_X$ which solves the integral equation (6). It suffices to calculate each term of the integral equation (6).

Let $(S_0(t))_{t\geq 0}$ be the integrated semigroup on $X$ generated by $A_0$. In view of the definition of $(S_0(t))_{t\geq 0}$, we have

$$\left((\lambda I - A_0)^{-1} x\right)(a) = \lambda \int_0^{+\infty} e^{-\lambda t} (S_0(t)x)(a)\, dt.$$

On the other hand, solving the equation

$$(\lambda I - A_0)\, y = x, \qquad \text{where } \lambda > 0,\ y \in D(A_0) \text{ and } x \in X,$$

we obtain

$$\left((\lambda I - A_0)^{-1} x\right)(a) = y(a) = \int_0^a e^{-\lambda t} x(a-t)\, dt.$$

Integrating by parts one obtains

$$\left((\lambda I - A_0)^{-1} x\right)(a) = e^{-\lambda a} \int_0^a x(t)dt + \int_0^a e^{-\lambda t} \left(\int_{a-t}^a x(s)ds\right) dt.$$

By uniqueness of Laplace transform, we obtain

$$(S_0(t)x)(a) = \begin{cases} \int_0^a x(s)\,ds, & \text{if } a \le t, \\ \int_{a-t}^a x(s)\,ds, & \text{if } a \ge t. \end{cases}$$

Using Proposition 14, we obtain

$$(S(t)\varphi)(\theta, a) = \begin{cases} \int_\theta^0 \varphi(s,a)\,ds + \int_0^a \varphi(0,\tau)\,d\tau, & \text{if } a \le t+\theta, \\ \int_\theta^0 \varphi(s,a)\,ds + \int_{a-t-\theta}^a \varphi(0,\tau)\,d\tau, & \text{if } a \ge t+\theta \ge 0, \\ \int_\theta^{t+\theta} \varphi(s,a)\,ds, & \text{if } t+\theta \le 0. \end{cases}$$

The assumption $\varphi(0,0) = 0$ implies that $S(\cdot)\varphi \in \mathcal{C}^1([0,T], \mathcal{C}_X)$ and we have

$$\frac{d}{dt}(S(t)\varphi)(\theta, a) = \begin{cases} 0, & \text{if } a \le t+\theta, \\ \varphi(0, a-t-\theta), & \text{if } a \ge t+\theta \ge 0, \\ \varphi(t+\theta, a), & \text{if } t+\theta \le 0. \end{cases}$$

Remark that the condition $\varphi(0,0) = 0$ is necessary to have $S(\cdot)\varphi \in \mathcal{C}^1([0,T], \mathcal{C}_X)$.

Let $G : [0,T] \to X$ $(T > 0)$ be a Bochner-integrable function and consider the function $K : [0,T] \to \widetilde{\mathcal{C}}_X$ defined by

$$K(t) = \int_0^t \widetilde{S}_0(t-s) X_0 G(s)\,ds.$$

For $a \le t+\theta$, we have

$$\begin{aligned} K(t)(\theta,a) = &\ \int_0^{t+\theta} (S_0(t+\theta-s)\, G(s))(a)\,ds, \\ = &\ \int_0^{t+\theta-a} \left(\int_0^a G(s)(\tau)\,d\tau\right) ds + \int_{t+\theta-a}^{t+\theta} \left(\int_{a-t-\theta+s}^a G(s)(\tau)\,d\tau\right) ds. \end{aligned}$$

For $a \ge t+\theta \ge 0$, we obtain

$$K(t)(\theta,a) = \int_0^{t+\theta} \left(\int_{a-t-\theta+s}^a G(s)(\tau)\,d\tau\right) ds,$$

and for $t+\theta \le 0$, we have $K(t)(\theta,a) = 0$.

The derivation of $K$ is easily obtained

$$\frac{dK}{dt}(t)(\theta,a)=\left\{\begin{array}{ll}\int_0^a G(\tau-a+t+\theta)(\tau)\,d\tau, & a\le t+\theta,\\ \int_{a-t-\theta}^a G(\tau-a+t+\theta)(\tau)\,d\tau, & a\ge t+\theta\ge 0,\\ 0, & t+\theta\le 0.\end{array}\right.$$

If we consider the function $u:[-r,T]\to X$ defined by

$$u(t,\cdot)=\left\{\begin{array}{ll} v(t)(0,\cdot), & \text{if } t\ge 0,\\ \varphi(t,\cdot), & \text{if } t\le 0.\end{array}\right.$$

Corollary 20 implies that $u_t=v(t)$. Hence, $u$ is the unique solution of (35).

The second part of Theorem 56 follows from Corollary 20 ■

**Theorem 57** *Assume that $F$ is continuously differentiable and there exist constants $L,\beta,\gamma\ge 0$ such that*

$$\begin{array}{l}|F(t,\varphi)-F(t,\psi)|\le L\,\|\varphi-\psi\|\,,\\ |D_tF(t,\varphi)-D_tF(t,\psi)|\le\beta\,\|\varphi-\psi\|\,,\\ |D_\varphi F(t,\varphi)-D_\varphi F(t,\psi)|\le\gamma\,\|\varphi-\psi\|\,.\end{array}$$

*Then, for given $\varphi\in\mathcal{C}_X$ such that*

$$\begin{array}{l}\dfrac{\partial\varphi}{\partial t}\in\mathcal{C}_X,\quad \varphi(0,\cdot)\in\mathcal{C}^1\left([0,l]\,,I\!R\right),\\ \varphi(0,0)=\dfrac{\partial\varphi}{\partial t}(0,0)=0\quad \textit{and}\\ \dfrac{\partial\varphi}{\partial t}(0,a)+\dfrac{\partial\varphi}{\partial a}(0,a)=f(0,a,\varphi(\cdot,a)),\ \textit{for } a\in[0,l]\,,\end{array}$$

*the solution $u$ of Equation (35) is continuously differentiable on $[0,T]\times[0,l]$ and is equal to the unique solution of Problem (34).*

**Proof.** One can use Theorem 22 (all the assumptions of this theorem are satisfied).

**Example 2**

$$\left\{\begin{array}{ll}\dfrac{\partial u}{\partial t}(t,x)=\Delta u(t,x)+f(t,x,u_t(\cdot,x)), & t\in[0,T]\,,\ x\in\Omega,\\ u(t,x)=0, & t\in[0,T]\,,\ x\in\partial\Omega,\\ u(\theta,x)=\varphi(\theta,x), & \theta\in[-r,0]\,,\ x\in\Omega,\end{array}\right. \tag{36}$$

where $\Omega\subset I\!R^n$ is a bounded open set with regular boundary $\partial\Omega$, $\Delta$ is the Laplace operator in the sense of distributions on $\Omega$ and $\varphi$ is a given function on $C_X:=C([-r,0]\,,X)$, with $X=\mathcal{C}\left(\overline{\Omega},I\!R\right)$.

Problem (36) can be reformulated as an abstract semilinear functional differential equation

$$\begin{cases} V'(t) = A_0 V(t) + F(t, V_t), & t \in [0, T], \\ V(0) = \varphi, \end{cases}$$

with

$$\begin{array}{l} D(A_0) = \left\{ u \in \mathcal{C}\left(\overline{\Omega}, I\!R\right); \quad \Delta u \in \mathcal{C}\left(\overline{\Omega}, I\!R\right) \text{ and } u = 0 \text{ on } \partial\Omega \right\}, \\ A_0 u = \Delta u, \end{array}$$

and $F : [0, T] \times \mathcal{C}_X \to X$ is defined by

$$F(t, \varphi)(x) = f(t, x, \varphi(\cdot, x)) \quad \text{for } t \in [0, T], \ \varphi \in \mathcal{C}_X \text{ and } x \in \Omega.$$

We have $\overline{D(A_0)} = \left\{ u \in \mathcal{C}\left(\overline{\Omega}, I\!R\right); \ u = 0 \ \text{ on } \partial\Omega \right\}, \neq X$.
Moreover

$$\begin{cases} \rho(A_0) \subset (0, +\infty) \\ \left\| (\lambda I - A_0)^{-1} \right\| \leq \frac{1}{\lambda}, & \text{for } \lambda > 0, \end{cases}$$

this implies that $A_0$ satisfies (**HY**) on $X$ (with $M_0 = 1$ and $\omega_0 = 0$).

Using the results of Section **??**, we obtain the following theorems (the all assumptions are satisfied).

**Theorem 58** *Assume that $F$ is continuous on $[0, T] \times \mathcal{C}_X$ and satisfies a Lipschitz condition*

$$|F(t, \varphi) - F(t, \psi)| \leq L \|\varphi - \psi\|, \quad t \in [0, T] \text{ and } \varphi, \psi \in \mathcal{C}_X,$$

*with $L \geq 0$ constant. Then, for a given $\varphi \in \mathcal{C}_X$, such that*

$$\Delta\varphi(0, \cdot) = 0, \ \text{on } \partial\Omega,$$

*there exists a unique integral solution $u : [0, T] \to X$ of the partial differential equation (36), i.e.*
*(i) $u \in \mathcal{C}_X$, $\Delta\left(\int_0^t u(s, \cdot) ds\right) \in \mathcal{C}\left(\overline{\Omega}, I\!R\right)$ and $\int_0^t u(s, \cdot) ds = 0$, on $\partial\Omega$,*
*(ii)*

$$u(t, x) = \begin{cases} \varphi(0, x) + \Delta\left(\int_0^t u(s, x)\, ds\right) + \int_0^t f(s, x, u_t(\cdot, x)\, ds, & \text{if } t \in [0, T], \text{ and } x \in \overline{\Omega} \\ \varphi(t, x), & \text{if } t \in [-r, 0], \text{ and } x \in \overline{\Omega}. \end{cases}$$

**Theorem 59** *Assume that $F$ is continuously differentiable and there exist constants $L, \beta, \gamma \geq 0$ such that*

$$\begin{array}{l} |F(t,\varphi) - F(t,\psi)| \leq L\,\|\varphi - \psi\|\,, \\ |D_t F(t,\varphi) - D_t F(t,\psi)| \leq \beta\,\|\varphi - \psi\|\,, \\ |D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)| \leq \gamma\,\|\varphi - \psi\|\,. \end{array}$$

*Then, for given $\varphi \in \mathcal{C}_X$ such that*

$$\begin{array}{l} \dfrac{\partial \varphi}{\partial t} \in \mathcal{C}_X, \quad \Delta\varphi(0,\cdot) \in \mathcal{C}\left(\overline{\Omega}, I\!R\right), \\ \varphi(0,\cdot) = \frac{\partial \varphi}{\partial t}(0,\cdot) = 0 \quad \text{on } \partial\Omega \text{ and} \\ \dfrac{\partial \varphi}{\partial t}(0,x) = \Delta\varphi(0,x) + f(0,x,\varphi(\cdot,x)), \ \text{ for } x \in \overline{\Omega}. \end{array}$$

*There is a unique function $x$ defined on $[-r,T] \times \Omega$, such that $x = \varphi$ on $[-r,0] \times \Omega$ and satisfies Equation (36) on $[0,T] \times \Omega$.*

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